On Fundamental Bifurcations from a Hysteresis Hyperchaos Generator

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Abstract—In this paper we discuss a four-dimensional autonomous circuit which includes one hysteresis element. This circuit is governed by two symmetric three-dimensional linear equations which are connected to each other by hysteresis switchings. Then we can derive the two-dimensional return map and show the following novel results: 1) Fundamental bifurcation diagram from periodic attractor to hyperchaos. It includes coexistence of torus and periodic attractor; 2) two-parameters bifurcation diagram. It exhibits some regularities for the onset of some periodic attractors and tori; 3) laboratory measurements of the return map attractors; 4) a sufficient condition for the existence of attractors.

I. INTRODUCTION

N order to understand chaotic circuit theory, it is important to consider the following problems: theoretical evidence for chaos, classification of chaos (e.g., classification by fractal dimension [3]) and route to chaos. These problems have been considered reasonably well for three dimensional circuits [4]–[7], but it is hard to analyze these problems in higher dimensional circuits.

This paper considers these problems in a simple four-dimensional circuit of Fig. 1. In this figure, the left op-amp. realizes a linear negative conductor characterized by \( i_2 = -g v_2 \). It is basically a three-segment voltage controlled conductor and we use only its central region. Also, the right op-amp. realizes a three-segment piecewise linear resistor characterized by

\[
\begin{align*}
\nu(i) &= -g_i i, \quad \text{for } |i| < \frac{B_i E}{r_i R_i}, \\
&= \frac{-B_i E}{r_i R_i}, \quad \text{for } i \geq \frac{B_i E}{r_i R_i}, \\
&= \frac{-B_i E}{r_i R_i}, \quad \text{for } i \leq -\frac{B_i E}{r_i R_i}.
\end{align*}
\] (1)

In this paper, we focus on the case where the small serial inductor \( L_0 \) is shorted (\( L_0 \rightarrow 0 \)). In this case, \( v_1 \rightarrow v \) and the above resistor becomes a hysteresis resistor characterized by

\[
\begin{align*}
i(v_1) &= \left\{ \begin{array}{ll} 
\frac{1}{r_1} (v_1 + (1 + \frac{B_1 E}{r_1 R_1})E), \quad & \text{for } v_1 \geq -E, \quad (2-1) \\
\frac{1}{r_1} (v_1 - (1 + \frac{B_1 E}{r_1 R_1})E), \quad & \text{for } v_1 \leq E, \quad (2-2)
\end{array} \right.
\end{align*}
\] (2)

where \( i \) is switched from (2–1) to (2–2) if \( v_1 \) hits the left threshold \(-E\) and vice versa. A theoretical evidence for this hysteresis behavior is discussed in [1]. Then the circuit dynamics are described by

\[
\begin{align*}
C_1 \frac{d v_1}{d t} &= -i_L - i(v_1), \\
C_2 \frac{d v_2}{d t} &= i_L + g v_2, \\
L \frac{d i_L}{d t} &= v_1 - v_2 - r_a i_L.
\end{align*}
\] (3)

Since \( i(v_1) \) is symmetric piecewise linear hysteresis, the dynamics are described by two symmetric three-dimensional linear equations which are connected to each other by the hysteresis switchings. It enables us to derive the two-dimensional return map. Then we focus on some parameter range and show the following novel results:

1) Fundamental route from periodic attractor to hyperchaos. It includes coexistence of torus and periodic attractor or that of torus and torus. They are confirmed by Lyapunov exponents [3] which are calculated by using a fast algorithm of piecewise exact solutions. Here, hyperchaos is usually defined as chaotic attractor with more than one positive Lyapunov exponent and is basic to classify chaos [8].

2) Two-parameters bifurcation diagram. It exhibits some regularities for the onset of some odd-periodic solutions and tori.

3) Laboratory measurements of the return map attractors.

4) A sufficient condition for the existence of attractors. The condition covers almost all phenomena discussed in (1) to (3).
In [1] and [2], we have considered some four dimensional hysteresis circuits. Reference [1] gives a rough bifurcation diagram from a periodic attractor to hyperchaos. However, its numerical simulations do not use the fast algorithm and coexistence of attractors is not discussed. Also, laboratory measurements of the return map attractors are not shown. Reference [2] gives some mathematical evidence for chaos whose 2-D Lyapunov exponent is positive. However, it focuses on the case where the chaotic dynamics is similar to the three-dimensional case.

There are some interesting results on more than three dimensional chaotic circuits [9]-[12]. However, these results are mainly based on numerical simulations. The discussion of bifurcation is not sufficiently systematic and more laboratory experiments are needed.

II. CIRCUIT AND RETURN MAP

In this section, we transform (3) into the Jordan form and derive the return map. Also, we use a fast calculation algorithm of the return map is given. First, we normalize (3). Using the following dimensionless variables and parameters:

\[ \gamma = \frac{1}{C_1 R_1}, \quad X_1 = \frac{u_1}{E}, \quad X_2 = \frac{u_2}{E}, \quad X_3 = \frac{u_3}{E}, \quad 2\sigma = r_1, \quad \rho = \frac{C_1 L}{r_1}, \quad \gamma = \frac{\gamma^2}{C_2}, \quad \eta = \frac{\eta^2}{C_2}, \quad \xi = \frac{\xi^2}{r_1}. \]

Equation (3) is transformed into the following dimensionless form:

\[ \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2\gamma & \gamma \\ \rho & -\rho & \rho \xi \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} - \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} h(X_1) \]  

where "," denotes the differentiation by \( \tau \), and a normalized hysteresis characterized by

\[ h(X_1) = \begin{cases} 1, & \text{for } X_1 \geq 1 \\ -1, & \text{for } X_1 \leq 1 \end{cases} \]

\( h(X_1) \) is switched from 1 to -1 if \( X_1 \) hits the left threshold -1 and vice versa. In this paper, we focus on the parameter range such that the coefficient matrix of (5) has real eigenvalue \( \Lambda \) and complex eigenvalues \( \Delta \pm j\omega \), where both \( \Lambda \) and \( \Delta \) are positive. By using the transformation:

\[ X = A^{-1}Bu, \quad \tau' = \omega \tau, \]

\[ X \equiv (X_1, X_2, X_3)^T, \quad u \equiv (u_1, u_2, u_3)^T, \quad A \equiv \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 - \rho & \rho & 1 + \rho \xi \end{bmatrix}, \quad B \equiv \begin{bmatrix} \Delta & 0 & 1 \\ 0 & 1 & \Lambda \\ \Delta^2 - \omega^2 & -2\Delta \omega & \Lambda^2 \end{bmatrix} \]

Equation (5) is transformed into the Jordan form:

\[ \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} \delta & 1 & 0 \\ -1 & \delta & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} h(z) \]

where \( z = u_1 + u_3, \quad \delta = \Delta - \lambda, \quad \lambda = \Delta \), and \( (q_1, q_2, q_3)^T = B^{-1}A(Q_1, Q_2, Q_3)^T \). Also, "," denotes the differentiation by \( \tau' \) and we use the symbol \( \tau' \) instead of \( \tau \) hereafter. Note that (7) is two symmetric linear equations on the following two half spaces which are connected to each other by \( h \).

\[ S_+ \equiv \{ (u, h) \mid u_1 + u_3 \geq -1, \ h = 1 \}, \quad S_- \equiv \{ (u, h) \mid u_1 + u_3 \leq 1, \ h = -1 \}. \]

In order to derive the return map, we define some objects in \( S_+ \) (see Fig. 2). Note that we omit "\( h = 1 \)" in the following:

\[ q = (q_1, q_2, q_3): \quad \text{Equilibrium point, } E^r = \{ u \mid u_1 = q_1, u_2 = q_2, u_3 = q_3 \geq -1 \}; \quad \text{Eigenspace corresponding to the real eigenvalue, } E^c \equiv \{ u \mid u_3 = q_3, u_1 + u_3 \geq 1 \}; \quad \text{Eigenspace corresponding to the complex eigenvalues, } B_- \equiv \{ u \mid u_1 + u_3 = -1 \}; \quad \text{Threshold plane, } B_+ \equiv \{ u \mid u_1 + u_3 = 1 \}; \quad \text{Symmetric object of } B_-, \quad B_+ \equiv \{ u \mid u_1 + u_3 = 1 \}; \quad \text{Domain of the map } D_- \equiv \{ u \mid \delta(u_1 - q_1) + (u_2 - q_2) + \lambda(u_3 - q_3) < 0, u_1 + u_3 = -1 \}; \quad \text{Decrease region of } u_1 + u_3 \text{ on } B_- \]

Note that the vector field in \( S_- \) is symmetric to that in \( S_+ \). Letting points on \( D_+ \) and \( D_- \) be represented by their \( u_1 \) and \( u_2 \) coordinates, we consider the trajectory starting from \( u_0 \equiv (u_{10}, u_{20}) \) on \( D_+ \) (see Fig. 2). Since \( \lambda \) and \( \delta \) are positive, this trajectory rotates divergently round \( E^r \) and it must hit \( D_+ \) at some positive time \( T_+ \). Let \( u_1 \equiv (u_{11}, u_{12}) \) denote this hit point. At this moment, it jumps onto the same point in \( S_- \). Since the vector field in \( S_- \) is symmetric to that in \( S_+ \), the trajectory starting from \( u_1 \) on \( D_- \) in \( S_- \) is symmetric to the trajectory starting from \( u_1 \) on \( D_-' \), where
Fig. 3. Bifurcation phenomena. Left column: horizontal = v, 1 v/div, vertical = r₁, 1 v/div. Center column: horizontal = v, 0.5 v/div, vertical = r₁, 0.5 v/div. (a) 1-periodic attractor, r₁ = 7.21kΩ, (σ = 0.545, μ₁ = -4.1 × 10⁻⁴, μ₂ = -8.2 × 10⁻⁴); (b) 1-torus, r₁ = 7.50kΩ, (σ = 0.55, μ₁ = 0.0, μ₂ = -1.3 × 10⁻³); (c) 7-periodic attractor, r₁ = 7.50kΩ, (σ = 0.5625, μ₁ = -1.1 × 10⁻³, μ₂ = -2.4 × 10⁻⁴); (d) 7-torus, r₁ = 7.67kΩ, (σ = 0.565, μ₁ = 0.0, μ₂ = -1.8 × 10⁻⁴).

D₁ is the symmetric object of D₂ in S₂. Thus we can define the following two-dimensional return map T:

\[ T : D₂ \rightarrow D₁, \quad u₀ \rightarrow -u₁. \]  

(8)

In Section IV, we give a sufficient condition for existence of attractors in this mapping. This return map can be calculated by the following:

\[
\begin{bmatrix}
    u₁ - q₁ \\
    u₂ - q₂ \\
    -1 - q₁ - q₃
\end{bmatrix} = \begin{bmatrix}
    f₁(τₛ, u₁₀, u₂₀) \\
    f₂(τₛ, u₁₀, u₂₀) \\
    f₃(τₛ, u₁₀, u₂₀)
\end{bmatrix} = e^{qτₛ} \begin{bmatrix}
    cosτₛ & sinτₛ & 0 \\
    -sinτₛ & cosτₛ & 0 \\
    cosτₛ & sinτₛ & e^{(λ-μ)τₛ}
\end{bmatrix} \begin{bmatrix}
    u₁₀ - q₁ \\
    u₂₀ - q₂ \\
    1 - u₁₀ - q₃
\end{bmatrix},
\]

(9)

where the switching time τₛ is given by solving the third row and the image of T is obtained by substituting this τₛ into the first and second rows. Then we can apply the following fast algorithm to find τₛ. It must catch the switching time τₛ. The problem is to find the solution of the following-type equation (fast algorithm):

\[ q(τₛ) = f(τₛ) + Ce^{λτₛ} + D = 0, \]  

(10)

where \( f(τₛ) = Ae^{8τₛ} \sin(τₛ + B), C < 0 \) and \( g(0) > 0 \). Here, A, B and C correspond to arbitrary constants determined in (9) and D corresponds to \( 1 + q₁ + q₃ \). Then we use the following algorithm, using the following steps:
Fig. 3. Continued. (e) chaotic attractor $r_1 = 7.78k\Omega, (\sigma = 0.576, \mu_1 \approx 2.5 \times 10^{-2}, \mu_2 \approx 1.7 \times 10^{-2})$, (f) hyperchaos $r_1 = 7.88k\Omega, (\sigma = 0.5765, \mu_1 \approx 2.5 \times 10^{-2}, \mu_2 \approx 4.4 \times 10^{-2})$, (g) hyperchaos $r_1 = 7.91k\Omega, (\sigma = 0.585, \mu_1 \approx 1.0 \times 10^{-1}, \mu_2 \approx 1.8 \times 10^{-1})$; (h) 1-periodic attractor $r_1 = 7.91k\Omega, (\sigma = 0.59, \mu_1 \approx -6.0 \times 10^{-1}, \mu_2 \approx -1.2)$.

1) Let $\tau_0 = 0$, $\tau_{n+1} = \tau_n + 2\pi$ and let $f(\tau_n)$ be local minimum for non-negative integer $n$. Then find $\tau_n$ such that $g(\tau_n - k) > 0$ for $1 \leq k \leq n$ and $g(\tau_n) < 0$.

2) Apply Newton-Raphson method from $\tau_{n-1}$. If the sequence of $\tau$ by Newton-Raphson method goes out of $[\tau_{n-1}, \tau_n]$, then go to STEP 3. Otherwise we obtain the solution.

3) Apply regular falsi in $[\tau_{n-1}, \tau_n]$.

Also, the Jacobian matrix $DT$ of $T$ is given by

$$-DT = \frac{\partial(f_1, f_2)}{\partial(u_{10}, u_{20})} - (\frac{\partial f_2}{\partial u_0})^{-1}(f_1, f_2)^T(\frac{\partial^2 f_1}{\partial u_0^2}, \frac{\partial^2 f_2}{\partial u_0^2}),$$  \hspace{1cm} (11)$$

where $f_i = f_i(\tau, u_{10}, u_{20}), i = 1, 2, 3$ in the above equation. We can calculate Lyapunov exponents by using (11) with the algorithm in [14]. The attractor is usually classified into the following by one- and two-dimensional Lyapunov exponents $\mu_1$ and $\mu_2$. If the attractor is stable periodic point then $\mu_1 < 0$ and $\mu_2 < 0$. If the attractor is torus then $\mu_1 = 0$ and $\mu_2 < 0$. If $\mu_1 > 0$ then the attractor is chaotic. And if $\mu_2 > 0$ and $1 - \mu_1 > 0$ then the attractor is hyperchaos [9].

III. FUNDAMENTAL BIFURCATION

Fig. 3 (route to chaos) shows fundamental bifurcation phenomena observed from the circuit. In the laboratory measurements, we choose $r_1$ as the control parameter: $7.2k\Omega < r_1 < 8k\Omega$. Other parameters are fixed as the following:

$L = 500mH, \quad g^{-1} = 6.3k\Omega, \quad C_1 = 63nF, \quad C_2 = 48nF, \quad r_0 = 380\Omega, \quad R_2 = 12.9\Omega, \quad R_3 = 10k\Omega, \quad V_o = 2.8v.
The left column of Fig. 3 shows projections of the trajectories and the central column shows attractors from the return map. The return map attractor can be observed by inputting the hysteresis switching pulses into the brightness control terminal (z-axis) of the oscilloscope. Here, n-periodic point implies that \( n \)-times composite of \( T \) (ab. \( T^n \)) gives a fixed point and \( n \)-torus implies that \( T^n \) gives single invariant closed curve. For numerical simulation of the right column, \( r_1 \) is the control parameter (value of \( \sigma \) is shown in the figure caption instead of \( r_1 \). \( \sigma \) is proportional to \( r_1 \)) and other parameters are fixed as the following:

\[
L = 500\text{mH}, \quad g^{-1} = 6.5\text{k}\Omega, \quad C_1 = 65nF, \quad C_2 = 46nF, \\
r_a = 350\Omega, \quad R_2 = 12.4\Omega, \quad R_1 = 10k\Omega.
\]

\( (\gamma = 1.354, \eta = 1.24, 0.54 \leq \sigma \leq 0.59, 6.21 \leq \rho \leq 7.41) \)

Also, Lyapunov exponents \( \mu_1 \) and \( \mu_2 \) are shown in the figure caption. Fig. 4 shows the bifurcation diagram which is obtained by plotting \( X_2 (x, y, z) \) coordinate of attractors for the case of increasing \( \sigma \) and Fig. 5 shows the corresponding Lyapunov exponents. In these numerical simulations, parameter condition is the same as Fig. 3 and the last value of \( X \) for some \( \sigma \) is used as the initial value for new \( \sigma \). In Fig. 3, laboratory measurements well coincide with numerical simulations at most 10% error. As \( r_1 \) increases, we have observed the following route:

1) For \( r_1 = 7.21k\text{\Omega} \), 1-periodic point is observed (Fig. 3(a)).
2) As \( r_1 \) increases, 1-periodic point changes to 1-torus via Hopf bifurcation (Fig. 3(h)). We can confirm that eigenvalues of \( DT \) cross the unit circle.
3) 1-torus becomes large and sharp. Then it changes to 7-periodic point like Fig. 3(c). Note that this is not due to phase locking, because the 7-periodic point is not located on the closed curve on which 1-torus existed. Around this parameter, we have confirmed coexistence of 1-torus and 7-periodic point. Its detail is explained afterward.
4) This 7-periodic point changes to 7-torus via Hopf bifurcation (Fig. 3(d)).
5) 7-torus becomes large and sharp. Then it changes to 28-periodic point via phase locking, because we confirm that this periodic point is located almost on the closed curve on which 7-torus existed.
6) This 28-periodic point changes to 56-periodic point and then to chaos like Fig. 3(e). This is due to period doubling. Then we can confirm that 1-D Lyapunov exponent becomes positive for \( \sigma > 0.574 \).
7) This chaotic attractor suddenly expands like Fig. 3(f) for \( r_1 = 7.5k\text{\Omega} \). This chaotic attractor satisfies \( \mu_2 > \mu_1 \) and is hyperchaos.
8) The hyperchaos disappears suddenly and 1-periodic point like Fig. 3(h) can be observed. It seems to be due to a kind of crisis [15].

Note that Lyapunov exponents converge reasonably after about 3000 iterations and we use from \( T^{-5000} \) to \( T^{-15000} \) to calculate them.

Fig. 4. Bifurcation diagram for increasing \( \sigma \).

Fig. 5. Lyapunov exponents for increasing \( \sigma \), \( \mu_1 \)=1-D exponent, \( \mu_2 \)=2-D exponent.

Fig. 7 (coexistence of attractors) shows the bifurcation diagram for both increasing and decreasing \( \sigma \). It is obtained by plotting \( X_2 (x, y, z) \) coordinate of attractors. For increasing \( \sigma \), the last value of \( X \) for some \( \sigma \) is used as the initial value for new \( \sigma \) and vice versa. In this figure, 1-torus and 7-periodic point co-exist for \( \sigma_A < \sigma < \sigma_B \). Fig. 8 gives its implication as the following:

1) The 1-torus is observed (Fig. 8-1).
2) As \( \sigma = \sigma_A = 0.56009 \), the 7-periodic point is born. Also, it is unstable for \( \sigma < \sigma_A \). For \( \sigma_A < \sigma < \sigma_B = 0.562368 \), this 7-periodic point and the 1-torus co-exist like Fig. 8-2. The stability is checked by Lyapunov exponents.
3) As \( \sigma \) increases, the domain of attraction [16] (ab. DA) for the 7-periodic point enlarges and it enters into the 1-torus (see Fig. 9).
Fig. 7. Enlargement of box 1 in Fig. 4. (a) Increasing \( \sigma \); (b) decreasing \( \sigma \).

Fig. 8. Sketch of coexistence of attractors, \( (\gamma = 1.354, \eta = 1.24) \). DA \( = \) domain of attraction.

Fig. 9. The domain of attraction for 7-periodic point. (a) \( \sigma = 0.5603 \), the closed curve implies 1-torus; (b) \( \sigma = 0.5612 \); (c) \( \sigma = 0.5623 \).

4) For \( \sigma = \sigma_B \), DA for the 7-periodic point collides with the 1-torus, thus the 1-torus disappears.

5) The 7-periodic point changes to 7-torus via Hopf bifurcation (Fig. 8-4).

If the collision of DA occurs after the Hopf bifurcation, we can observe the coexistence of 7-torus and 1-torus. Its concrete example is shown in Fig. 10.

Fig. 11 (two-parameters bifurcation diagram) shows rough 2-parameters bifurcation diagram. This figure is obtained by using Lyapunov exponents. The control parameters are \( r_1 \) and \( R_2 \). \( \nu \) and \( \eta \) are proportional to \( r_1 \) and \( R_2 \), respectively, and are indicated in the figure instead of \( r_1 \) and \( R_2 \). Note that odd-periodic point and odd-torus can be regularly observed. Fig. 12 shows some odd-torus. If we vary \( \sigma \) near odd-periodic point existence regions then we can observe the following route to hyperchaos: 1-periodic point \( \rightarrow \) (Hopf bifurcation) \( \rightarrow \) 1-torus \( \rightarrow \) (Jump due to coexistence of attractors) \( \rightarrow \) odd-periodic point \( \rightarrow \) (Hopf bifurcation) \( \rightarrow \) odd-torus \( \rightarrow \) (phase locking) \( \rightarrow \) periodic point \( \rightarrow \) (period doubling) \( \rightarrow \) chaos \( \rightarrow \) hyperchaos \( \rightarrow \) 1-periodic point.

IV. ATTRACTOR EXISTENCE CONDITION

This section gives a sufficient condition which guarantees the existence of some subset \( D_s \) in the domain of \( T \) such that \( T(D_s) \subset D_s \). First, we introduce the linear system from the
Fig. 12. Some odd-tori: Scale as in Fig. 3, except for center of (a) = 1 v x 1 v (a) 3-torus, \( r_1 \approx 12.6k\Omega, R_5 \approx 48.3k\Omega \) \( (\eta = 4.6, \sigma = 0.99) \); (b) 5-torus, \( r_1 \approx 8.66k\Omega, R_2 \approx 17.2k\Omega \) \( (\eta = 1.68, \sigma = 0.6295) \); (c) 9-torus, \( r_1 \approx 7.38k\Omega, R_2 \approx 10.5k\Omega \) \( (\eta = 1.06, \sigma = 0.5398) \); (d) 11-torus, \( r_1 \approx 7.28k\Omega, R_2 \approx 10.3k\Omega \) \( (\eta = 0.97, \sigma = 0.526) \).

Jordan form (7):

\[
\begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3
\end{bmatrix} =
\begin{bmatrix}
\delta & 1 & 0 \\
-1 & \delta & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\begin{bmatrix}
u_1 - q_1 \\
u_2 - q_2 \\
u_3 - q_3
\end{bmatrix}
\] (12)

and define some objects (see Fig. 13): \( D_+ \equiv \{u|u_1 + u_3 = 1, u_3 < q_3\} \) \( D_- \equiv \{u|\delta(u_1 - q_1) + (u_2 - q_2) + \lambda(u_3 - q_3) < 0, u_1 + u_3 = -1\} \): Decrease region of \( u_1 + u_3 \) on the plane \( u_1 + u_3 = -1 \), \( L_2 \equiv \{u|\beta(u_1 - q_1) + (u_2 - q_2) + \lambda(u_3 - q_3) = 0, u_1 + u_3 = -1\} \): Boundary between the decrease and the increase of \( u_1 + u_3 \) on \( D_- \). \( M_1 \equiv \{u|\beta(u_1 - q_1) + (u_2 - q_2) = 0\} \): Boundary between the decrease and the increase of \( u_1 \). \( L_\alpha \equiv \{u|u_3 = -1 - q_1\} \): (It includes \( E_r \cap D_- \)) \( L_+ = \{u|u_1 = 2 + u_1\} \): (It includes \( L_\alpha \cap D_+ \)) Then let \( L'\), \( L'_2 \) and \( L'_+ \) be symmetric to objects of \( L_2, L_\alpha \) and \( L_+ \), respectively. For this linear system, the objective problem can be translated into the existence of some subset \( D_s \) in \( D_+ \) such that any trajectory starting from \( D_s \) must hit its symmetric object \( D'_s \) in \( D_- \). Then we introduce two key trajectories: The first key trajectory \( u^* \) starts from the line \( L'_1 \cap M_1 \) at \( \tau = 0 \) and it hits the line \( L'_+ \cap L_\alpha \) at some positive time \( \tau_c \) up to \( \frac{\pi}{2} - \phi_{u^*} \), where \( \phi_{u^*} = \tan^{-1}\delta \). Note that \( u^* \) is unique and \( \tau_c \) can be calculated easily. Moreover, \( u^* \) must hit \( D_- \) at some positive time \( \tau_D \) such that \( \tau_c < \tau_D < 1.5\pi - \phi_{u^*} \). Based on \( u^* \), we define the subset \( D_s \) in \( D_1 \) as the following: \( L_D \equiv \{u|u_1 = u^*_1(\tau_D)\} \); \( L'_D \equiv \) The symmetric object of \( L_D \). \( T_u \equiv \{u|(u_1, u_2) \in (u^*_1(\tau), u^*_2(\tau)), u_1 \geq -u^*_1(\tau_D))\} \); \( D_s \equiv \)
Area in \(D_t\) surrounded by \(T_+, L_+, L_0^+\) and \(L_0^+ \cap D_+\), where \((u^*_0(\tau), u^*_2(\tau))\) is the locus of \(u_1\) and \(u_2\) components of \(u^*\) for \(0 < \tau < 2\pi\). Then we assume

\[
u^*_3(\tau_D) > -q_3. \quad (C1)
\]

The second key trajectory \(*u\) starts from the plane \(M_1\) at \(\tau = 0\) and it passes through the point \(L_0^+ \cap M_1 \cap L_x\) at \(\tau = \pi\). Note that \(*u\) is also unique. Then let \(T_0 \equiv \{ (u_1(\tau), u_2(\tau)) : 0 < \tau < 1.5\pi - \phi_3 \}\); also, let \(u(0)\) be an initial point on \(D_s\) and let \(*u\) hit \(D_-\) at \(\tau = \pi\). Then we have:

**Theorem:** 1) We assume \((C1)\). For any \(u(0)\) on \(D_s\), \(\tilde{u}_3(\tau_s) > u^*_3(\tau_D)\) is satisfied if

\[
u^*_3(0) < 1 + q_1 \quad \text{and} \quad *u_1(0) + *u_2(0) < -1. \quad (C2)
\]

2) Let \(BG\) be the left side line segment of \(D_s\) \(BG \equiv D_s \cap L_D^+\) (see Fig. 13). For any \(*u(0)\) on \(D_s\), \(\tilde{u}_3(\tau_s) < 1 + q_1\) is satisfied if

\[
\tilde{u}_3(\tau_s) < 1 + q_1, \quad \text{for} \quad \tilde{u}_3(0) \in BG. \quad (C3)
\]

3) Let \(u_0 = g(u_1, u_3)\) be the top arc of \(D_s\) and let \(E \equiv (E_1, E_2, E_3)\) be the product of \(u^*\) and \(L_x^+\) for \(0 < \tau < 2\pi\). For any \(*u(0)\) on \(D_s\), \(\tilde{u}_2 < g(u_1(\tau_s), u_3(\tau_s))\) is satisfied if

\[
u^*_3(\tau_D) > B_3, \quad -E_3 < u^*_2(\tau_s). \quad (C4)
\]

where \(B_3\) is the \(u_2\) component of \(B\). That is, \((C1)\) to \((C4)\) guarantees \(\tilde{u}_3(\tau_s) \in D_s^+\) for any \(*u(0)\) \(\in D_s\) and therefore they guarantee \(T(D_s) \subset D_s\). Note that \((C1)\) to \((C4)\) are given rigorously by implicit inequalities described by only 5 parameters \((\lambda, \delta, q_1, q_2, q_3)\). Fig. 14 shows a projection of parameter region which satisfies \((C1)\) to \((C4)\). Note that this region includes the bifurcation diagram of Fig. 11.

**Proof:** We show only for 1): First we define some objects on \(L_x\) (see Fig. 13): \(\alpha \equiv \text{The line segment that connects } E_\alpha \) and \(u^*(\tau_\alpha)\), \(D_A \equiv \text{The closed area surround by } T_u, T_v, \alpha \) and \(L_0^-\), where \(L_0^- \equiv \{ (u_1 = q_1, u_2 < q_2) \}\), \(D_B \equiv \text{The closed area surrounded by } T_u, T_v, \alpha \) and \(L_0^+\), where \(L_0^+ \equiv \{ (u_1 = q_1, u_2 > q_2) \}\), \(D_C \equiv \text{The closed area surround by } T_u \) and \(L_0\). Noting that \((C1)\) to \((C3)\) guarantee any trajectory starting from \(D_s\) does not go out of \(T_u\) by the time when it hits \(D_-\). The problem is to show that any trajectory starting from \(D_s\) must hit \(D_+\) by the time when it intersects \(x = 1\) and because it does not intersect any trajectory starting from \(x = -1\). Letting \(u^*\) be a trajectory starting from \(x = 0\), \(u^*\) hits \(D_-\) by the time when it intersects \(T_u\) and because it does not intersect any trajectory starting from \(x = -1\). Also, \((C2)\) guarantees any trajectory which passes through \(D_0\) at 0 is under \(D_-\) at some negative time when it intersects \(M_1\) and therefore it is on \(D_-\) at the negative time. Moreover, any trajectory starting from \(D_C\) must hit \(D_-\) by \(\tau = \pi\) and the hit point is upper \(u^*_3(\tau_D)\).

\[\text{Q.E.D.}\]

**V. CONCLUSION**

For four-dimensional hysteresis circuits, we have confirmed some route from periodic attractor to hyperchaos. It includes coexistence of torus and periodic attractors. And we have concluded them in a two-parameters bifurcation diagram in which odd-periodic point and odd-torus were regularly generated. Some of numerically confirmed phenomena were reproduced by laboratory measurements. Also we have proved a sufficient condition for existence of attractors. In order to develop these
results more generally, we should consider the following: 1) Analysis of the nonconstrained case \( L_0 > 0 \); 2) classification of chaos; 3) rigorous analysis of bifurcations.

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REFERENCES


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