A 4-D manifold piecewise linear hyperchaos generator
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Abstract—This paper proposes a 4-D autonomous piecewise linear system and gives a sufficient condition for generation of three kinds of chaos: line-expanding chaos, area-expanding chaos and hyperchaos. An implementation example is also demonstrated.

I. INTRODUCTION
This paper proposes a 4-D manifold piecewise linear system (ab.MPL) that is a developed version of 3-D MPL [11][12]. As shown in Fig.1, the 4-D MPL consists of 2-D linear system, 1-D linear system and a controlled source \( u \) whose output corresponds to the fourth state. \( u \) takes two values \( \pm 1 \) and is switched from \(-1\) to \(1\) (respectively \(1\) to \(-1\)) if the states satisfies \( y = 0 \) and \( z \geq z \) (respectively, \( y = 0 \) and \( x < z \)). Form this system, we can derive 2-D return map that is non-invertible and is exactly piecewise linear. As the main result, we give a sufficient conditions for generation of three kinds of chaotic attractors; line-expanding chaos, area-expanding chaos and hyperchaos [1]. An implementation example is also proposed and hyperchaos generation is verified.

Higher dimensional autonomous chaotic systems have been studied with great interests [2]-[9]. Such systems can exhibits higher dimensional chaos represented by hyperchaos that can not be observed in 3-D systems. They relates important basic problems: classification of chaos, recognition of chaos, route to chaos and so on. Also, active application of higher dimensional chaos has begun to emerge. e.g., hyperchaos-based communication [10]. However, analysis of such system is hard because of complex nonlinearity and simple model is desired in order to approach higher dimensional chaos. Note that the 4-D MPL is the first system in which hyperchaos generation can be proved rigorously. Our previous 4-D and 5-D piecewise linear models [3][4] have been investigated by using numerical analysis based on the piecewise exact solutions.

II. CIRCUIT AND RETURN MAP
The 4-D manifold piecewise linear system (ab.MPL) is described by

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 25 & 0 & 0 \\
0 & 0 & \lambda & -K/\lambda \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix} \cdot u(x, y, z, w),
\]

where

\[
D_a = \{(x, y, z, w)|w = 1\},
D_b = \{(x, y, z, w)|w = -1\},
\]

\( w \in \{-1, 1\} \) is a dummy variable in order to describe the switching behavior and \( \epsilon \) denotes differentiation by normalized r. Letting \( Th_a = \{(x, y, z, w)|x > z, y = 0, w = 1\} \) and \( Th_b = \{(x, y, z, w)|x < z, y = 0, w = -1\} \), \( u \) is switched from \(-1\) to \(1\) if the solution on \( Da \) hits \( Th_a \) and is switched from \(-1\) to \(1\) if the solution on \( Db \) hits \( Th_b \).

Note that switchings occur only at the moment when the state satisfies \( y=0\). Also, this equation has real characteristic root \( \lambda \) and complex characteristic root \( \delta + j\omega \), \( \omega \ll \delta \). Here, the three parameters are assumed to be

\[-1 < \delta < 1, \quad -1 < \lambda < 1, \quad K > 0.\]  (2)

Fig.2 shows an implementation example. Here, \(-g\) and \(-g_1\) are linear negative conductors. The switching terminals (a) and (b) correspond to \( u = 1 \) and \( u = -1 \), respectively. In this figure, \( i, v \) and \( VI \) are proportional to \( z, y \) and \( z \), respectively. Using the following dimensionless variables and parameters

\[
\tau = \Omega t, \quad x = \frac{r}{E} i, \quad \epsilon = \frac{r}{E_k} v_1, \quad \delta = \frac{\Delta}{\Omega}, \quad \lambda = \frac{A}{\Omega}, \quad \Lambda = \frac{1}{C_1}(g_1 - \frac{r_1}{\epsilon}),
\]

\[
K = \frac{K_s}{\Omega}, \quad K_s = \frac{1}{C_1 R_s r_1}, \quad \Omega = \sqrt{\frac{1}{LC}},
\]

the circuit equation is transformed into Equation (1). The dependent switch \( S_w \) is switched from (a) to (b) if \( v_1 > R_s i \) and \( v = 0 \), and is switched from (b) to (a) if \( v_1 \leq
Fig. 2 Implementation example

In order to derive the return map, we also define

\[
A_a \equiv \{(x, y, z) \mid z \leq x, y = 0\} \subseteq D_2, \\
A_b \equiv \{(x, y, z) \mid z > x, y = 0\} \subseteq D_2, \\
A \equiv A_a \cup A_b.
\]

Noting that trajectories started from \(A\) at \(r = 0\) must return to \(A\) at \(r = \frac{\pi}{w}\), we can define the following two-dimensional map from \(A\) to itself:

\[
F: A \rightarrow A, \quad (x_0, z_0) \mapsto (x_1, z_1), \tag{5}
\]

where \((x_0, z_0)\) and \((x_1, z_1)\) are a started point and a hit point, respectively. We can calculate this mapping rigorously:

\[
(x_1, z_1) = F(x_0, z_0) = (f(x_0, z_0), g(x_0, z_0)), \tag{6}
\]

where

\[
f(x_0, z_0) = \begin{cases} 
  +e_2x_0 + (e_2+1) & \text{for } z_0 \leq x_0, \\
  -e_2x_0 - (e_2+1) & \text{for } x_0 > z_0,
\end{cases}
\]

\[
g(x_0, z_0) = \begin{cases} 
  e_{2\pi}z_0 + (e_{2\pi}+1)K & \text{for } z_0 \leq x_0, \\
  e_{2\pi}z_0 - (e_{2\pi}+1)K & \text{for } x_0 > z_0.
\end{cases}
\]

Now the system dynamics is simplified into piecewise linear iteration that is characterized by three parameters \(\delta, \lambda, \text{ and } K\):

\[
(x_1, z_1) = F(x_0, z_0), \quad \text{for } (x_0, z_0) \in A. \tag{7}
\]

Typical waveform and two-dimensional plots are shown in Fig. 4.

III. ANALYSIS OF RETURN MAP

First, we focus on \(\delta > 0\) and \(\lambda > 0\), in this case: \(|\frac{\partial f}{\partial z_0}| > 1\) and \(|\frac{\partial g}{\partial z_0}| > 1\) are guaranteed on \(A\), and the return map is stretching almost all directions. Hence it is sufficient to prove hyperchaos generation that \(F\) has an attractor such that \(F(S) \subseteq S\), where we define hyperchaos as an attractor form the 2-D return map that is stretching almost all directions. This definition can guarantee positiveness of two Lyapunov exponents if they exist. However, rigorous proof for the existence of the exponents is hard [14] and we will discuss it elsewhere. In Fig. 3, hyperchaos generation is guaranteed theoretically in the shaded region. That is derived in the next section.

If \(F\) has an attractor, we can give the following. (see Fig. 3)

1. If \(\delta > 0\) and \(\lambda > 0\), \(F\) exhibits hyperchaos.
2. If \(\delta + \lambda > 0\), \(F\) exhibits area-expanding chaos.
3. If \(\delta > 0\) or \(\lambda > 0\), \(F\) exhibits line-expanding chaos.
4. If \(\delta < 0\) and \(\lambda < 0\), \(F\) has periodic orbit.

In Fig. 3, the attractor existence region is given by numerical experiments.

IV. MAIN RESULT

Here, we derive an attractor existence condition that can guarantee hyperchaos generation. We introduce some object as the following. First, \(CS_+\) is a corn surface in \(D_+\) determined uniquely by an initial point \((x(0), y(0), z(0))\) [15] Any trajectory started from \(CS_+\) moves on \(CS_+\) until it hits \(Th_+\). We obtain

\[
CS_+(x(0), y(0), z(0)) = \left\{ (x, y, z, w) \right\} \\
(x-1)^2 + \left(\frac{1}{w} (y-\delta (x-1))\right)^2 = r(0)^2 e^{2ks},
\]

\[
z + \frac{K}{\lambda} = e^{4s}(z(0) + \frac{K}{\lambda}), -\infty < s < \infty, w = 1
\]

where \(r(0)^2 \equiv (x(0) - 1)^2 + \left(\frac{1}{w} (y(0) - \delta (x(0)) - 1))\right)^2\) and \(z(0) > -\frac{K}{\lambda}\).
Second, the interior of \(CS_+\) is given by
\[
CV_+(x(0), y(0), z(0)) \equiv \left\{ (x, y, z, w) \mid \begin{array}{l}
(x-1)^2 + \left( \frac{1}{\omega} (y - \delta (x-1)) \right)^2 \leq r(0)^2 e^{2s_+} \\
z + \frac{K}{\lambda} = e^{\lambda t}(z(0) + \frac{K}{\lambda}), -\infty < s < \infty, w = 1
\end{array} \right\},
\]
where \(r(0)^2 \equiv (x(0) - 1)^2 + \left( \frac{1}{\omega} (y(0) - \delta (x(0) - 1)) \right)^2 \) and \(z(0) > -\frac{K}{\lambda}\).

Third, the product of \(Cl_j\) and \(y=0\) is given by
\[
\alpha_+(x(0), y(0)) \equiv CV_+(x(0), 0, z(0)) \cap \{(x, y, z, w) | y = 0\}.
\]
Let \(CS_-\), \(CV_-\) and \(\alpha_-\) be symmetric object of \(CS_+\), \(CV_+\) and \(\alpha_+\), respectively. Note that \(F\) maps any point \((x_0, z_0) \in A_+ \cap \alpha_+\) into \(\alpha_+\), because any trajectory starting from \(CV_+\) moves in \(CV_+\) until it hits \(T_+\):
\[
F(\alpha_+ \cap A_+) \subset \alpha_+.
\]
Next, let \(T_{\theta 0}\) be the edge of \(T_+\) and let \((p, p) = F(T_{\theta 0}) \cap T_0\). They are given by
\[
T_{\theta 0} = \{(x, z)|x = z\}, \quad f_+^{-1}(p) = g_+^{-1}(p).
\]
Using the intersection \((p, p)\), we fix \(\alpha_+\) as the following.
\[
\alpha_+(x(0), y(0)) \equiv CV_+(x(0), 0, z(0)) \cap \{(x, y, z, w) | y = 0\}.
\]
Let \(CS_-\), \(CV_-\) and \(\alpha_-\) be symmetric object of \(CS_+\), \(CV_+\) and \(\alpha_+\), respectively. Note that \(F\) maps any point \((x_0, z_0) \in A_+ \cap \alpha_+\) into \(\alpha_+\), because any trajectory starting from \(CV_+\) moves in \(CV_+\) until it hits \(T_+\):
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\[
T_{\theta 0} = \{(x, z)|x = z\}, \quad f_+^{-1}(p) = g_+^{-1}(p).
\]
Using the intersection \((p, p)\), we fix \(\alpha_+\) as the following.
\[
\alpha_+(x(0), y(0)) \equiv CV_+(x(0), 0, z(0)) \cap \{(x, y, z, w) | y = 0\}.
\]

Finally, we give some definitions for the iteration \((z_{n+1}, z_n) = F(z_n, z_n)\). Let \(F^n \equiv (f^n, g^n)\) denote \(n\)-fold composition of \(F\) and let \(F^{-n} \equiv (f^{-n}, g^{-n})\) denote \(n\)-fold composition of the inverse of \(F\).

As the following, we make basic regions \(S_+\) and \(T_+\) on \(D_+\):
\[
S_+ \equiv \alpha_+(p, -p) \cap \{(x, z)|x \leq L_+(z), z < x\}.
\]
\[
T_+ \equiv \alpha_+(p, -p) \cap \{(x, z)|x \leq L_+(z), z < x\}.
\]
where \(L_+(z) = f_+ \circ g_+^{-1}(z)\) and \(L_+(z) = f_+ \circ g_+^{-1}(z)\).

Note that \(g_+^{-1}(z_1) \leq f_+^{-1}(x_1)\) (respectively, \(g_+^{-1}(z_1) \leq f_+^{-1}(x_1)\)) is satisfied if \((x_0, z_0) \in A_+\) (respectively, \(g_+^{-1}(z_0) \leq f_+^{-1}(x_0)\)). Noting (11), the following is satisfied.
\[
F(S_+) \subset S_+ \cup T_+.
\]
Letting \(S_-\) and \(T_-\) be symmetric object of \(S_+\) and \(T_+\), respectively, (14) guarantees \(F(S_-) \subset S_- \cup T_-\) because the system is symmetric. Therefore \(T_- \subset S_+\) guarantees \(F(S_- \cup S_-) \subset S_+ \cup S_-\) and existence of attractor. Then we have

**Theorem 1:** Existence of attractor and hyperchaos generation are guaranteed if
\[
x_b \geq M_+(z_b), x_0 \leq N_+(z_b), x_c \leq N_+(z_c),
\]
\[
z_c \leq L_+(x_c), z_2 > -\frac{K}{\lambda}.
\]

where, \((x_0, z_0) = (f_+(p, -p), g_+(p, -p)), (x_b, z_b) = (f_+(p, -p), g_+(p, -p)), (x_c, z_c) = (f_+(x_c), g_+(x_c))\).

(Proof) Noting that \(z = M_+(x)\) and \(z = N_+(x)\) are monotone functions for \(z > -\frac{K}{\lambda}\), Condition (15) guarantees
\[
M_+(z) \leq L_+(z), \quad z_0 \leq z \leq -q,
\]
\[
N_+(z) \geq L_+(z), \quad z_0 \leq z \leq z_0,
\]
\[
M_+(z) \leq L_+(z), \quad z_0 \leq z \leq q.
\]
where \(L_+(z)\), \(L_+(z)\) and \(M_+(z)\) are symmetric objects of \(L_+(z)\), \(L_+(z)\) and \(M_+(z)\), respectively. Condition (16) guarantees \(T_- \subset S_+\).

Condition (15) is described by only three parameters \((\delta, \lambda, K)\) and is satisfied in the shaded region in Fig.3.

**IV. Conclusion**

We have proposed 4-D manifold piecewise linear system and have given by a sufficient condition for hyperchaos, area-expanding chaos and line-expanding chaos. Now we are considering 1) route to hyperchaos, 2) classification of higher dimensional chaos, and 3) controlling hyperchaos and its engineering applications.

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**REFERENCES**


Fig.4 Laboratory measurement ($C = 0.047 \mu F, C_1 = 0.1 \mu F, L = 300 mH, E = 0.05 V$)