A SIMPLE NONAUTONOMOUS CHAOTIC CIRCUIT
WITH A PERIODIC PULSE-TRAIN INPUT

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ABSTRACT

This paper presents a simple nonautonomous circuit with a periodic pulse-train input. If the input does not exist, the circuit exhibits either equilibrium attractor or hard oscillation. If a periodic pulse-train input is applied, the circuit exhibits interesting chaotic and periodic attractors. The circuit behavior can be analyzed by a mapping procedure based on a 1-D return map focusing on the moments when the input is applied. Using a simple test circuit, we can verify typical phenomena in the laboratory.

1. INTRODUCTION

This paper presents a simple nonautonomous circuit consisting of an RLC resonator and an impulsive switch controlled by a state comparator and a periodic pulse-train input. If the input does not exist, the circuit exhibits either equilibrium attractor or hard oscillation. If an equidistant pulse-train input is applied, the circuit can exhibit chaotic attractors. That is, the RLC resonator is changed into a chaos generator by the operation of the impulsive switch. The RLC resonator is the second order circuit, and the dynamics of the impulsive switch corresponds to the third state. Therefore, our model can be regarded as a 3-D nonautonomous system. This 3-D nonautonomous system can be analyzed by a mapping procedure based on a 1-D return map focusing on the moments when the input is applied. This mapping procedure is valid in the case of a non-equidistant pulse-train input. If a non-equidistant pulse-train input is applied, the circuit behavior can be analyzed by a composite of different 1-D return maps corresponding to different pulse-intervals. We show typical chaotic and periodic attractors in this case. Using a simple test circuit, we can verify these phenomena in the laboratory.

It should be noted that our circuit relates to resonate-and-fire neurons, a novel neuron model for which chaotic behavior has not been reported [1]. Our nonautonomous circuit is based on autonomous chaotic circuits with impulsive switchings [2, 3]. For the autonomous version we have given a theoretical evidence for chaos generation and have clarified bifurcation phenomena. Using the autonomous chaotic circuits, a chaotic pulse-coupled network (CPCN) can be constructed. The CPCN can exhibit rich chaos synchronous phenomena, and relates to pulse-coupled neural networks (PCNNs, [4, 5]). The PCNNs consist of relaxation (non-chaotic) oscillators and is applicable to image processing [6, 7], associative memory [8, 9], and so on. Analysis of the circuits with the impulsive switchings is important not only as basic nonlinear problems but also for engineering applications. We believe that this paper contributes to develop study of switched chaotic dynamics and its engineering applications.

2. CIRCUIT MODEL

Figure 1 shows the objective circuit. The periodic pulse-train input \( U(t) \) consists of two pulse-intervals \( T_1 \) and \( T_2 \).

\[
U(t) = \begin{cases} 
V_H, & \text{at } t = n(T_1 + T_2) \text{ or } t = n(T_1 + T_2) + T_1, \\
V_L, & \text{otherwise},
\end{cases}
\]

where \( n \) is non-negative integer. If the switch \( S \) is opened all the time, the circuit behaves as an RLC resonator.

\[
C \frac{dv}{dt} = i, \quad L \frac{di}{dt} = -v - Ri.
\]

The switch \( S \) is closed impulsively if either \( v \) reaches the threshold \( V_T \) (self-switching) or the pulse signal arrives \( (U(t) = V_H, \text{ compulsory-switching}) \). At the moment when \( S \) is closed, \( v \) is reset to the base \( E \) instantly holding \( i = \text{constant} \).

\[
v(t') = E, \quad \text{if } v(t) = V_T \text{ or } U(t) = V_H.
\]

We focus on the case where Equation (2) has complex characteristic roots \( -\delta \omega \pm j\omega \).

\[
\delta \omega = \frac{R}{2L} > 0, \quad \omega^2 = \frac{1}{LC} - \left( \frac{R}{2L} \right)^2 > 0.
\]

In this case, \( v \) can vibrate below the threshold \( V_T \). Using the dimensionless variables and parameters

\[
\tau = \omega t, \quad x = \frac{v}{V_T}, \quad y = \frac{\dot{x}}{\dot{v}}, \quad q = \frac{E}{V_T},
\]

\[
u(\tau) = \frac{1}{\nu_T} \left( U(t) - V_L \right), \quad d_1 = \omega T_1, \quad d_2 = \omega T_2,
\]

Equations (1) to (3) are transformed into Equations (4) to (6).

\[
\frac{d}{d\tau} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\delta & 1 \\ -1 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{if } x(\tau) < 1 \text{ and } u(\tau) = 0 . \quad (4)
\]

\[
\begin{bmatrix} x(\tau^+) \\ y(\tau^+) \end{bmatrix} = \begin{bmatrix} q \\ y(\tau) - \delta(x(\tau) - q) \end{bmatrix}, \quad \text{if } x(\tau) = 1 \text{ or } u(\tau) = 1 . \quad (5)
\]
where theoretically. This case can be regarded as a kind of hard oscillation.

As a preparation of the analysis we consider the circuit without the input \( u(\tau) = 0 \). For \( \delta > 0 \), the origin is the stable equilibrium point and the circuit exhibits equilibrium attractor\(^1\). If a periodic attractor exists, it must coexist with the equilibrium attractor and the circuit exhibits either depending on the initial state (see Figure 2). This case can be regarded as a kind of hard oscillation. Figure 3 shows existence region of each attractor which is given theoretically [10].

\(^1\)If the resistor \( R \) is replaced with a linear negative resistor \(-R\), \( \delta \) becomes negative. For \( \delta < 0 \), the origin is the unstable equilibrium point and the circuit can exhibit chaos [2].

In this section, we consider responses of the circuit to the periodic pulse-train input \( (u(\tau) \neq 0) \). In order to analyze the phenomena, we define a 1-D return map. First, we consider the case \( d \equiv d_1 = d_2 \), that is, the pulse-interval of the input is equidistant. Figure 4 shows the phase plane, where we define \( B = \{(x, y, \tau) \mid x = q, \tau = 0 \mod d\} \). Let a point on \( B \) be represented by its \( y \)-coordinate. As a trajectory starts from a point \( y_0 \in B \) at \( \tau = 0 \), the compulsory-switching occurs at \( \tau = d \) and the trajectory is reset to a point \( y_1 \in B \). Then we can define the following 1-D return map.

\[ H_d : B \rightarrow B, \quad y_0 \mapsto y_1, \quad y_1 = H_d(y_0). \]  

Note that the 1-D return map \( H_d \) is characterized by the pulse-interval \( d \). For \( 0 < \tau < d \), several self-switchings may occur. However, focusing on the compulsory-switching moments, the dynamics can be analyzed by the 1-D return map. This 1-D return map is given analytically by using Equations (5) and (7). Let \( DH_d(y_0) \equiv \frac{\partial}{\partial y_0} H_d(y_0) \). Lyapunov exponent of the 1-D return map (8) is given by

\[ \lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |DH_d(y_n)|. \]  

Figures 5(a) to (c) show typical 1-D return maps and attractors for the equidistant pulse-train input \( (d \equiv d_1 = d_2) \). For simplicity, we fix the parameters \( \delta = 0.01 \) and \( q = -0.8 \) in the numerical simulations. In this case the circuit exhibits only equilibrium attractor if the input does not exist \( (u(\tau) = 0) \). In Figure 5(a), a stable fixed point \( y_p \) appears in the 1-D return map and the phase space attractor is periodic by the only compulsory-switching. Figures 5(b) and (c) show chaotic attractors where both the self- and compulsory-switchings occur. Each Lyapunov exponent of the 1-D return map is positive. That is, the RLC resonator is changed into the chaos generator by the operation of the impulsive switch. For chaos generation, local instability is required. Our circuit can exhibit the self-switching by occurrence of the compulsory-switching. As shown in Figure 4, the self-switching can expand the initial deviation \( \Delta y_0 \) to the deviation \( \Delta y_1 (> \Delta y_0) \) at the next
compulsory-switching moment. It causes the expanding parts in the 1-D return maps.

Next, we consider the case $d_1 \neq d_2$, that is, the pulse-interval of the input is non-equidistant. In this case, the dynamics can be described by a composite of the 1-D return map for the pulse-interval $d_1$ and that for the pulse-interval $d_2$:

$$H \equiv H_{d_1} \circ H_{d_2} : B \to B, \quad y_0 \mapsto y_1,$$

where $y_1 = H(y_0) = H_{d_2}(H_{d_1}(y_0))$. It should be noted that the dynamics can be described by a composite of different 1-D return maps corresponding to different pulse-intervals even if an input pulse-train consists of various pulse-intervals. Lyapunov exponent of the 1-D return map (10) is given by

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left( \ln |DH_{d_1}(y_n)| + \ln |DH_{d_2}(H_{d_1}(y_n))| \right).$$

Figures 5(d) to (f) show typical 1-D return maps and attractors for the non-equidistant pulse-train input ($d_1 \neq d_2$). Each return map is a composite of two different 1-D return maps chosen from Figures 5(a) to (c). We can observe the following phenomena.

- Figure 5(d) shows a composite map of the two different chaotic 1-D return maps in Figures 5(b) and (c). A stable fixed point $y_p$ appears in the composite map and the phase space attractor is periodic.
- Figure 5(e) shows a composite map of the periodic 1-D return map in Figure 5(a) and the chaotic 1-D return map in Figure 5(b). A stable fixed point $y_p$ appears in the composite map and the phase space attractor is periodic.
- Figure 5(f) shows a composite map of the periodic 1-D return map in Figure 5(a) and the chaotic 1-D return map in Figure 5(c). The attractor is chaotic and the Lyapunov exponent of the composite map is positive. However, a stable fixed point $y_p$ exists in the composite map. Therefore, the phase space attractor can be either periodic or chaotic depending on the initial states.

### 4. LABORATORY EXPERIMENTS

In order to observe typical phenomena in the laboratory, we implement a test circuit as shown in Figure 6. In the figure, $R_l$ denotes an internal resistor of the inductor $L$. The inductor current is observed by the op-amp (TL074) and the resistor. The switch $S$ is realized by an analog switch (4066). The pulse-train for the self-switching is realized by the comparator (LM339) and the monostable multi-vibrator (4538). If the capacitor voltage $v$ reaches the threshold voltage $V_T$, the comparator triggers the mono-stable multi-vibrator to output the switching pulse. The input pulse-train (for the compulsory-switching) is realized by the pair of the comparators and the mono-stable multi-vibrators. A sine wave is input to each comparator. By each comparator output, each mono-stable multi-vibrator generates an equidistant pulse-train. If $V_{T_1} \neq V_{T_2}$, these pulse-trains have phase difference. Adding these pulse-trains gives the input pulse-train $U(t)$.

Figures 7(a) to (c) show typical attractors for the equidistant pulse-train input ($T \equiv T_1 = T_2$). Figures 7(d) to (f) show typical attractors for the non-equidistant pulse-train input ($T_1 \neq T_2$). Figures 7(a) to (f) correspond to the simulation data in Figures 5(a) to (f), respectively.
5. CONCLUSION

We have studied a simple nonautonomous chaotic circuit with a periodic pulse-train input. If the input does not exist, the circuit exhibits either equilibrium attractor or hard oscillation. Applying the input, the circuit exhibits chaotic and periodic phenomena. Using a mapping procedure, we have analyzed these phenomena. This mapping procedure will be a useful tool to analyze systems with pulse-train input(s). Using a simple test circuit, we have verified typical phenomena in the laboratory. In future works, we are considering (1) detailed analysis of bifurcation phenomena for wider parameter region, and (2) extension to some applications including time series prediction and pulse-based communications.

6. REFERENCES


Figure 7: Typical phenomena in the laboratory ($C \approx 1.5$ [nF], $L \approx 300$ [mH], $R_L \approx 260$ [Ω], $V_T \approx 1.0$ [V], $E \approx -0.8$ [V]), left column: phase space attractors, vertical=i, 50 [μA/div.], horizontal=v, 0.5 [V/div.], right column: time-domain waveforms, horizontal=t, 100 [μs/div.], vertical=v, $U$, 0.5 [V/div.]. (a), (b) and (c) responses to the equidistant pulse-train input. (d), (e) and (f) responses to the non-equidistant pulse-train input.