A BUCK-BOOST CONVERTER CONTROLLED BY PERIODIC INPUTS

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ABSTRACT

This paper presents a buck-boost converter controlled by periodic inputs and provides the basic theoretical results. As an efficient analysis tool, we derive a slope chart that displays relationship between slopes of the return map and those of the inputs. Applying the tool for the case of triangular inputs, we can give a sufficient condition for chaos generation. We then give a method to realize uniform distribution of the chaotic switching time intervals for any DC output: it might be a criterion to improve EMC. A method to realize super stable periodic behavior is also given.

1. INTRODUCTION

This paper presents a simplified model of the buck-boost converter (ab. SBBC [1]) whose switching can be controlled by two synchronized periodic inputs. The system dynamics is described by a switched dynamical system having a scalar state, where the two inputs are applicable to the lower and the upper switching thresholds, respectively. The vector field is switched discontinuously when the state hits either threshold. This switched system can describe a simplified model of the buck converter [2], [3], that of the boost converter [4], and other practical systems: a human sleep-waking model [5], a neural-type oscillator [6] and so on.

For the SBBC, we provide basic theoretical results. First, we derive a one dimensional return map and its derivative theoretically for arbitrary shapes of the inputs. Based on the derivative, we construct a slope chart that displays relationship between slopes of the return map and those of the inputs. It is a useful tool to elucidate the system stability. Applying the tool for the case of triangular inputs, we can give a sufficient condition for chaos generation [7] for any DC output. Then, adjusting the inputs appropriately, the uniform distribution of the chaotic switching time intervals can be achieved for any DC output. If the switching is chaotic, the EMC problem might be improved as suggested in [8], and the uniform distribution might be a criterion for the improvement. Developing the method into a sampled state feedback version of practical DC/DC converters, an efficient control might be realized.

In conventional systems [1]-[4], it is hard to realize the uniform distribution of the switching time intervals for any DC output. A robust method to realize super stable periodic behavior [9] is also given.

Fig. 1: Simplified model of the buck-boost converter

2. BUCK-BOOST CONVERTER

Fig. 1 shows a simplified model of the buck-boost converter (ab. SBBC). In this system, the switch S is controlled by two synchronized periodic current signals \( S_{m1} \) and \( S_{m2} \), respectively. The ideal diode operates as an inverse phase switch to S. Based on this switching rule, the dynamics is described by

\[
\frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{cases} V_1 & \text{if } S = \text{ON} (i < U_2(t)), \\ -V_2 & \text{if } S = \text{OFF} (i > U_1(t)), \end{cases}
\]

where \( 0 < V_2 < V_1 \) and \( 0 < V_1 < V_2 \) correspond to the buck and the boost modes, respectively. This simplification is based on the replacing an RC pair in the original system with a constant voltage source \( V_2 \) for \( RC \gg T \) [2]. The SBBC corresponds to a PWM control model [4] if \( U_2 \) is a constant and \( U_1 \) is a saw-tooth signal. Using the following dimensionless variables and parameters:

\[ \tau = \frac{t}{T}, \quad x = \frac{i - I_0}{I_0}, \quad l_1 = \frac{T V_1}{L I_0}, \quad l_2 = \frac{T V_2}{L I_0}, \]

where \( I_0 \) is a DC current such that \( U_1(t) < I_0 \leq U_2(t) \), Equation (1) is transformed into

\[
\dot{x} = \begin{cases} l_1 & \text{for } x < u_2(\tau), \\ -l_2 & \text{for } x > u_1(\tau), \end{cases}
\]

where \( \dot{x} \equiv \frac{dx}{d\tau}, u_1(\tau) = \frac{U_1(T \tau - l_0)}{l_0}, u_2(\tau) = \frac{U_2(T \tau - l_0)}{l_0}, \) and \( u_1(\tau + \frac{1}{12}) = u_1(\tau), \)

\( u_2(\tau + \frac{1}{12}) = u_2(\tau). \) Note that Equation (2) can describe a simplified model of the buck converter [2], [3], that of the boost converter [4], and other practical systems: a human sleep-waking model [5], a neural-type oscillator [6] and so on.
3. SLOPE CHART

As shown in Fig. 2, we consider the trajectory started from \( L_d \equiv \{(\tau, x)| x = 0\} \) with \( x = -l_2 \) at \( \tau = \tau_1 \). The trajectory decreases and hits the lower threshold \( u_1(\tau) \). Let \((y_1, u_1(y_1))\) be the hit point. Then \( x \) changes from \(-l_2\) to \( l_1\), and the trajectory increases and hits the upper threshold \( u_2(\tau) \). Let \((z_1, u_2(z_1))\) be the hit point. Then \( x \) changes from \( l_1 \) to \(-l_2\) at \( \tau = \tau_1 \), and the trajectory decreases and intersects \( L_d \). Let \((\tau_2, 0)\) be the intersection. We can define a one dimensional map for the intersection time as the following:

\[
\tau_{n+1} = f(\tau_n), \quad \tau_n \in R,
\]

\[
\begin{align*}
0 &= u_2(z_n) - l_2(\tau_{n+1} - z_n) \\
u_2(z_n) &= u_1(y_n) + l_1(z_n - y_n) \\
u_1(y_n) &= -l_2(y_n - \tau_n)
\end{align*}
\]

Introducing the restricted variable, \( \theta_n = \tau_n \pmod{1} \), we obtain the following return map:

\[
\theta_{n+1} = F(\theta_n) \equiv f(\theta_n) \pmod{1}, \quad \theta_n \in I \equiv [0, 1).
\]

Fig. 3 shows typical return maps. Differentiating Equation (3), we obtain the derivative of the return map:

\[
DF(\theta_n) \equiv \frac{d\tau_{n+1}}{d\tau_n} \frac{dz_n}{dy_n} \frac{dy_n}{d\tau_n} = \frac{(l_1 - \dot{u}_1(y_n))(l_1 + \dot{u}_2(z_n))}{(l_2 + u_1(y_n))(l_1 - u_2(z_n))}
\]

where \( \dot{u}_1(y_n) > -l_2 \) and \( \dot{u}_2(z_n) < l_1 \) must be satisfied in order to hit the lower and the upper thresholds, respectively. Using Equation (5), the derivative of the return map can be calculated using the derivatives of the inputs at the switching moments \((\dot{u}_1(y_n)\) and \(\dot{u}_2(z_n)\)).

In the numerical analysis, a periodic point can be given by calculating \( F^k(p) = p \) with Equation (3) and its stability can be checked using Equation (5), where \( F^k \) denotes the \( k \)-fold composition of \( F \). Equation (5) can also be used to calculate Lyapunov exponent of chaotic attractor. Hereafter, we refer to the derivative of the return map \( DF \) as the map-slope and refer to the derivatives of the inputs, \( \dot{u}_1 \) and \( \dot{u}_2 \), as the input-slopes.

Now we regard Equation (5) formally as a function of the two input-slopes. Letting \( a \equiv u_1 \) and letting \( b \equiv u_2 \), we define the function from \( R^2 \) to \( R \) as follows:

\[
DF(a, b) = \begin{cases} 
(l_1 - a)(l_2 + b) & \text{for } (a, b) \in D_r, \\
(l_2 + a)(l_1 - b) & \text{for } (a, b) \in D_i, \\
0 & \text{for } (a, b) \in D_o, \\
\end{cases}
\]

where \( DF(a, b) = 0 \) means that the image is empty. It corresponds to the case where the state \( x \) can not hit the threshold with such input-slope. In the domain of \( DF(a, b) \), we note the important borders:

\[
\begin{align*}
D_r &\equiv \{(a, b)| l_2 < a, l_1 < b\}, \\
D_i &\equiv \{(a, b)| l_2 \geq a \cup \{(a, b)| l_1 \geq b\}, \\
B_+ &\equiv \{(a, b)| a - b = 0\} \cap D_r, \\
B_0 &\equiv \{(a, b)| a = l_1 \cup \{(a, b)| b = l_2\} \cap D_r, \\
B_- &\equiv \{(a, b)| -(b - \frac{1}{2}(l_2 - l_1))(a - \frac{1}{2}(l_1 - l_2)) \leq \frac{1}{2}(l_1 + l_2)^2\} \cap D_r, \\
B_\infty &\equiv \{(a, b)| a - l_2 \cup \{(a, b)| b - l_1\}
\end{align*}
\]

Using these borders, we divide the domain into 10 subregions which characterize the map-slope:

\[
\begin{align*}
1 < DF(a, b) &\quad \text{for } (a, b) \in C_+, \\
0 < DF(a, b) < 1 &\quad \text{for } (a, b) \in S_1^1, \\
-1 < DF(a, b) < 0 &\quad \text{for } (a, b) \in S_2^1, \\
-1 < DF(a, b) < 0 &\quad \text{for } (a, b) \in S_2^1, \\
0 < DF(a, b) < 1 &\quad \text{for } (a, b) \in S_1^2, \\
DF(a, b) < -1 &\quad \text{for } (a, b) \in C_1^1 \cup C_2^2, \\
DF(a, b) = 0 &\quad \text{for } (a, b) \in D^0 \cup D^1 \cup D^2.
\end{align*}
\]
The borders and the subregions are illustrated in Fig. 4: this is the slope chart. Note that the subregions can be described analytically. In the slope chart, we can find the following basic properties.

(P1) The origin, \((a, b) = (0, 0)\), exists on the segment \(B_+\) and corresponds to an autonomous relaxation oscillator. The origin \(O\) exists on the lower half (respectively, upper half) of the segment \(B_+\) for \(l_2 > l_1\) (respectively, \(l_2 < l_1\)).

(P2) A segment on the a-axis (through \(O\)) or \(-b\)-axis corresponds to the case where either input is constant. If the first input-slope satisfies \(a_- < a < a_+\) and if the second input-slope is zero, \(DF(a, 0)\) for any \((a, 0)\in\{(a, b)|a_- < a < a_+, b = 0\}\) is a candidate of the map-slope.

(P3) A rectangle including \(O\) corresponds to the case where neither input is constant. If the two input-slopes satisfy \(a_- < a < a_+\) and \(b_- < b < b_+\), \(DF(a, b)\) for any \((a, b)\in\{(a, b)|a_- < a < a_+, b_- < b < b_+\}\) is a candidate of the map-slope. For a stable periodic orbit, a set of point(s) in the rectangle is available (see Fig. 3(b)). For chaotic attractor, it is pretty hard to identify available part(s) in the rectangle. This case might cause a flexible variety of the dynamics.

(P4) The slope chart is symmetric with respect to the line \(\{(a, b)|-b = \frac{1}{2}(l_2 - l_1) = a = \frac{1}{2}(l_2 - l_1)\}\): the boost mode is equivalent to the boost mode by exchanging \((l_1, l_2, a, -b)\) with \((l_2, l_1, -b, a)\) on the chart.

(P5) If the inputs are piecewise linear, the input-slopes are constants and the map-slopes are given as a set of point(s) on the slope chart. In this case, detailed analysis is possible as suggested in the next section.

The map-slope can give important information for the system stability. For example, if \(|DF(a, b)| > 1\) is satisfied on \(I\), chaos generation is guaranteed theoretically [7]. Note that observation of the slope chart is much simpler than analysis of the return map itself. Effects of the frequency and phase of the inputs will be discussed elsewhere.

Theorem 1, the SBBC generates chaos if the map-slopes satisfy \(|DF(a, b)| > 1\) on \(I\). Applying the slope chart to the following triangular inputs (see Fig. 5), we can provide some interesting theoretical results as suggested in (P5).

\[
u_1(\tau) = \begin{cases} 
-\alpha_2 \tau - \gamma_1 & \text{for } 0 \leq \tau < d_1, \\
\alpha_1 (\tau - \frac{1}{\alpha_1}) - \gamma_1 & \text{for } d_1 \leq \tau < \frac{1}{\alpha_1}, 
\end{cases}
\]

\[
u_2(\tau) = \begin{cases} 
\beta_2 \tau + \gamma_2 & \text{for } 0 \leq \tau < d_2, \\
-\beta_1 (\tau - \frac{1}{\beta_1}) + \gamma_2 & \text{for } d_2 \leq \tau < \frac{1}{\beta_1}, 
\end{cases}
\]

where \(d_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}\), \(d_2 = \frac{\beta_1}{\beta_1 + \beta_2}\), \(u_1(\tau + \frac{1}{\alpha_1}) = u_1(\tau),\ u_2(\tau + \frac{1}{\beta_1}) = u_2(\tau),\) and the parameters \((\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)\) are positive. Noting again that the SBBC exhibits chaos if the map-slopes satisfy \(|DF(a, b)| > 1\) on \(I\) [7], we give the following results.

**Theorem 1:** For the single input case, either input is zero, the SBBC generates chaos if Boost mode \((l_1 < l_2)\) with \(u_2(\tau) = 0\):

\((-\alpha_2, 0) \in C_+ \cup D^2,\ (\alpha_1, 0) \in C^1._{-}\),

Buck mode \((l_1 > l_2)\) with \(u_1(\tau) = 0\):

\((0, \beta_2) \in C_+ \cup D^1,\ (0, -\beta_1) \in C^2._{-}\).

Using the chaotic attractor, the EMC problem might be improved as suggested in [8]. In order to realize the improvement, we might find a key in the distribution of the switching time intervals given by

\[\Delta \tau_n \equiv \tau_{n+1} - \tau_n = f(\tau_n) - \tau_n.\]

We then have the following results which might be a criterion for the EMC improvement.

**Theorem 2:** For the boost mode \((l_1 < l_2)\) with \(u_2 = 0\), we assume \((-\alpha_2, 0) \in C_+\) and \((\alpha_1, 0) \in C^1._{-}\). As shown in Theorem 1, the SBBC generates chaos in this case and there exists an integer \(N\) such that \(1 \leq N - 1 < |DF(\alpha_1, 0)| \leq N\). Letting \(m_1 M\) be an integer such that \(2 \leq m_1 M \leq N\), there exist \(\alpha_1\) and \(\alpha_2\) such that

\[-DF(-\alpha_2, 0)DF(\alpha_1, 0) = (DF(-\alpha_2, 0) - DF(\alpha_1, 0))M\]

Since the slope chart is symmetric, the results for either mode is sufficient, however, we show both results for the emphasis.
If Equation (9) is satisfied, the return map has a uniform invariant measure and the distribution of the switching time interval $\Delta \tau_n$ is uniform on $[K_a, K_b + D_b]$, where $K_a = \gamma_1(\frac{1}{2} + \frac{1}{l_2})$, $D_b = M(1 - \frac{1}{DF(0, \beta_2)})$. Fig. 6(a) shows an example of the return map.

![Return Maps for Triangular Inputs](image)

Fig. 6: Return maps for triangular inputs ($l_1 = 0.5, l_2 = 1, \gamma_1 = 1, \alpha_1 = 6, \alpha_2 = 0.6, m_1 = 1$) (a) Chaotic map for uniform switching interval distribution ($u_2(\tau) = 0$). (b) Piecewise constant map ($\beta_1 = 1, \beta_2 = 1, \gamma_2 = 0, m_2 = 4$)

For the buck mode ($l_1 > l_2$) with $u_1 = 0$, we assume $(0, \beta_2) \in C_+$ and $(0, -\beta_1) \in C_2^-$. In this case there exists an integer $N$ such that $1 \leq N - 1 < |DF(0, -\beta_1)| \leq N$, and let $m_2 M$ be an integer such that $2 \leq m_2 M \leq N$. There exist $\beta_1$ and $\beta_2$ such that

$$-DF(0, \beta_2)DF(0, -\beta_1) = (DF(0, \beta_2) - DF(0, -\beta_1))M$$

(10)

If Equation (10) is satisfied, the distribution of $\Delta \tau_n$ is uniform on $[K_b, K_b + D_b]$, where $K_b = \gamma_2(\frac{1}{2} + \frac{1}{l_2})$, $D_b = M(1 - \frac{1}{DF(0, \beta_2)})$.

That is, for each mode, the appropriate triangular inputs can realize the uniform distribution independently of $l_1$ and $l_2$: the input $V_1$ and the output $V_2$. Also, the distribution range is adjustable by $\gamma_1$ or $\gamma_2$.

THEOREM 3: For the double input case, neither input is constant, the SBBC generates chaos if the inputs satisfy Table 1 and Table 2.

THEOREM 4: The map-slopes are constants almost everywhere (see Fig.6(b)) hence the SBBC exhibits super stable periodic behavior [9] if the input-slopes satisfy $(a, b) \in B_0$:

$$\beta_1 = l_2, \ \ \beta_2 > l_1, \ \ \text{for any } u_1(\tau), \ \ \text{or}$$

$$\alpha_1 = l_1, \ \ \alpha_2 > l_2, \ \ \text{for any } u_2(\tau).$$

Table 1: Chaos generating condition for boost mode

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Table 2: Chaos generating condition for buck mode

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5. CONCLUSION

In order to analyze basic dynamics from the SBBC, we have derived the slope chart that displays relationship between the input-slopes and the map-slopes. The chart enables us to give a sufficient conditions for chaos generation, to realize uniform distribution of chaotic switching time intervals, and to realize super stable periodic behavior. Developing the results into a sampled state feedback version of practical DC/DC converters, an efficient control might be realized. Also, the slope chart might be useful to clarify interesting bifurcation phenomena from switched dynamical systems [10]-[12].

6. REFERENCES