Abstract
This paper introduces a resonance phenomenon from an Integrate-and-Fire Model (ab. IFM) with plural periodic inputs. When two inputs are applied, the IFM usually outputs aperiodic pulse-trains. However, adjusting the parameters of an input, the IFM outputs a periodic pulse-train with a constant pulse interval, while the state is quasi-periodic. This resonance phenomenon is confirmed in the laboratory, and is analyzed using the mapping procedure.

1. Introduction
As the continuous-time neuron models, the Hodgkin-Huxley model [1], the BVP model [2] and various Integrate-and-Fire Models (ab. IFMs) [3]-[9] have been investigated intensively. The basic dynamics of the IFM is as the following: the state corresponds to the nerve membrane potential and is charged up by the stimulation. When the state reaches a firing threshold, the IFM fires: it outputs a firing pulse and the state is reset to a base level. The dynamics of the IFM with a single input has been deeply considered (periodic stimulation [3], periodic base [4], periodic threshold [3][5][9], and noisy stimulation [6][7]). Also, networks of IFMs have been considered in order to explore the brain function [8].

In this paper we consider an IFM to which two periodic inputs are applicable: one is additional stimulation input and the other is applied to the base. The IFM usually outputs aperiodic pulse-trains. However, adjusting the base input appropriately, the IFM exhibits an interesting resonance phenomenon: the output pulse-train is periodic with a constant pulse interval while the state is quasi-periodic. This periodic pulse-train may imply that the IFM recognizes the stimulation signal. We confirm the resonance phenomenon in the laboratory, and analyze it using the mapping procedure.

2. Integrate-and-Fire Model with Periodic Inputs
Fig.1 shows a circuit version of the Integrate-and-Fire Model (ab. IFM) to which a stimulation input \( S(t) \) and a base input \( B(t) \) are applicable. The capacitor voltage \( v \) corresponds to the nerve membrane potential and is charged up by the current source \( I_0 + S(t) > 0 \). When \( v \) reaches the firing threshold \( Th_0 \), the monostable multivibrator (ab. MM) outputs a firing pulse \( Y \) that resets \( v \) to the base \( B(t) \), where \( B(t) < Th_0 \). Repeating this manner, the IFM outputs a pulse-train. In this paper we consider the following shapes of the inputs:
\[
S(t) = K_s \sin(\frac{2\pi}{T}t), \quad B(t) = K_b \sin(\frac{2\pi}{T}(t - \Phi_b)). \tag{1}
\]

The circuit dynamics is described by
\[
\begin{align*}
\dot{x} &= s_o + s(\tau), \quad \text{for} \quad x < 1, \\
x(\tau^+) &= b(\tau^+), \quad \text{if} \quad x(\tau) = 1, \\
y(\tau^+) &= \begin{cases} 0, & \text{for} \quad x < 1, \\ 1, & \text{if} \quad x(\tau) = 1, \end{cases}
\end{align*}
\tag{2}
\]
where the following dimensionless variables and parameters are used:
\[
s(\tau) = k_s \sin(2\pi \tau), \quad b(\tau) = k_b \sin(2\pi (\tau - \Phi_b)),
\]
\[
x = \frac{v}{Th_0}, \quad y = \frac{Y - V_L}{V_H - V_L}, \quad \tau = \frac{t}{T}, \tag{3}
\]\[s_o = \frac{I_o T}{C Th_0}, \quad K_s = \frac{K_s T}{C Th_0}, \quad K_b = \frac{K_b}{Th_0}, \quad \Phi_b = \frac{\Phi_b}{T}.
\]
For simplicity, we assume
\[
s_0 > |k_s|, \quad s_0 > 2\pi |k_b|, \quad k_b < 1. \tag{4}
\]
Fig.2 illustrates the system dynamics, where \( \tau_n \) denotes the n-th pulse position.

We have implemented the IFM using an OTA (MJN13600), an MM (4538), a comparator (LM339) and a
switch (4066). Usually, the IFM behaves as in Fig.3(a): both the output $Y$ and the state $v$ are quasi-periodic. However, adjusting the parameters $B_K$ and $B_{\Phi}$ of the base, the IFM exhibits an interesting phenomenon as shown in Fig.3(b): the output pulse-train is periodic with a constant pulse interval, while the state is quasi-periodic. That is, the IFM shows a resonance phenomena of the output pulse interval as the stimulation and the base inputs are matched. In the following, we analyze such phenomena based on the mapping procedure.

3. Analysis

We first clarify the effect of each input and then consider the resonance phenomenon.

First, let us consider the case where the stimulation input is applied to the IFM, i.e., $s_k \neq 0$ and $b_k = 0$. Then the pulse positions are described as the following:

$$\tau_{n+1} = f_s(\tau_n) \equiv H^{-1}(H(\tau_n) + \frac{1}{s_0}),$$

where $H$ is given by

$$H(\tau) \equiv \int_{s_0}^{\tau} (s_0 + s(t))\,dt$$

$$= \tau + \frac{k_s}{2\pi s_0} (1 - \cos(2\pi \tau)) \cdot$$

We call the map $f_s$ pulse position map. Since $s_0 > k_s$, $H$ is monotone-increasing and satisfies

$$H(0) = 0, \quad H(\tau + 1) = H(\tau) + 1.$$  

Since the stimulation input $s(\tau)$ is periodic with period 1, we can obtain the return map:

$$\theta_{n+1} = F_s(\theta_n) \equiv f_s(\theta_n) \pmod{1},$$

$$F_s : I \to I, \quad I \equiv [0, 1).$$

Since $f_s$ is monotone-increasing and satisfies $f_s(\tau + 1) - f_s(\tau) = 1$, $F_s$ is a circle map as shown in Fig.4(a). In order to characterize the pulse-train dynamics, we consider the pulse interval $\Delta \tau_n \equiv \tau_{n+1} - \tau_n$. The density of the pulse interval is shown in Fig.4(a). The
average of the pulse intervals is given by

\[ \bar{\Delta \tau} \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \Delta \tau_i. \]  

(9)

This is identical to the rotation number \( \rho \), which is independent of the initial value \( \tau_1 \) \([3][10]\). Also, the orbit from \( F_s \) is ergodic if \( \rho \) is irrational, and is periodic if \( \rho \) is rational. Making the change of the variable \( \psi_n = H(\theta_n) \), we have \( \psi_{n+1} = \psi_n + 1/s_0 \); the rotation number of \( F_s \) is given by \( \rho_s = 1/s_0 \).

Next, let us consider the case where the base input is applied to the IFM, i.e., \( k_s = 0 \) and \( k_b \neq 0 \). Repeating a similar consideration to Equation (5), we obtain the pulse position map

\[ \tau_{n+1} = f_b(\tau_n) \equiv \tau_n + \frac{1}{s_0} (1 - b(\tau_n)) \]  

(10)

Since \( s_0 > 2\pi |k_b| \), \( f_b \) is monotone-increasing and \( f_b(\tau + 1) - f_b(\tau) = 1 \). Similar to Equation (8) we obtain the return map

\[ \theta_{n+1} = F_b(\theta_n) \equiv f_b(\theta_n) \quad (\text{mod} \ 1) \]  

(11)

which is a circle map as shown in Fig.4(b).

Here, we consider the case where both inputs are applied to the IFM, i.e., \( k_s \neq 0 \) and \( k_b \neq 0 \). We then obtain the pulse position map

\[ \tau_{n+1} = f_{sb}(\tau_n) \equiv H^{-1}(H(\tau_n) + \frac{1}{s_0} (1 - b(\tau_n))) \]  

(12)

and the return map

\[ \theta_{n+1} = F_{sb}(\theta_n) \equiv f_{sb}(\theta_n) \quad (\text{mod} \ 1). \]  

(13)

Fig.5(a) shows the return map, where the parameters \( s_0 \), \( k_s \), \( k_b \) and \( \phi_b \) correspond to the original parameters in Fig.3(a). Adjusting the amplitude \( k_s \) and the phase \( \phi_b \) of the base input, we obtain the return map in Fig.5(b). It corresponds to Fig.3(b) and we can see that the pulse interval density is changed into a delta-function: the output pulse-train becomes periodic with a constant pulse interval. That is, the IFM shows a resonance phenomena of the pulse interval.

For a given stimulation with the parameters \( s_0 \) and \( k_s \), the resonance phenomenon is given by the base input \( b(\tau) \) with the parameters

\[ k_b = \frac{k_s \sin(\pi / s_0)}{\pi}, \quad \phi_b = \frac{1}{2} - \frac{1}{2s_0}. \]  

(14)

Substituting (14) into (12) we obtain the linear pulse position map:
It guarantees that the pulse interval is always 
\[ \Delta \tau_n = 1/s_0 \] which is equal to the rotation number \( \rho_{sb} \).

If \( s_0 \) is irrational, the periodic output pulse-train is not synchronized with the inputs, i.e., the state is quasi-periodic. Note that rational \( s_0 \) is measure zero. Out of the resonance parameter values (14), the pulse position map is not linear as (15) but is usually nonlinear given by (12). Hence the IFM may output aperiodic or periodic pulse-trains as coupled relaxation oscillator [10].

3. Conclusions
We considered the IFM with plural periodic inputs, and clarified that the IFM shows a resonance phenomenon of the interval of the output pulse-train. The phenomenon was confirmed in the laboratory and analyzed using the mapping procedure. Now we are considering (1) detailed analysis of the resonance phenomena, (2) design of an IC version of the IFM, and (3) development of an IFM based artificial neural network.

References

\[ F_{sb}(\theta_n) \]

\[ \tau_n \]

\[ 0 \to 1 \]

\[ \Delta \tau_n \]

\[ 0 \to 1 \]

\[ 1 \]

\[ 30 \]

\[ 0 \]

\[ \theta_n \]

\[ 0 \to 1 \]

\[ F_{sb}(\theta_n) \]

\[ \Delta \tau_n \]

\[ 0 \to 1 \]

\[ 1 \]

\[ 30 \]

\[ 0 \]

\[ \theta_n \]

\[ 0 \to 1 \]

\[ F_{sb}(\theta_n) \]

\[ \Delta \tau_n \]

\[ 0 \to 1 \]

\[ 1 \]

\[ 30 \]

\[ 0 \]

\[ \theta_n \]

\[ 0 \to 1 \]

Fig. 5. Return map and pulse interval density from the IFM with stimulation and base inputs. \( s_0 = \sqrt{2.2} \cdot \phi_s = 0.4 \cdot \phi_n = 0.16 \cdot k_s = 0.74 \) and \( k_s = 0.2 \cdot (a) \phi_s = 0.4 \cdot (b) \phi_n = 0.16 \).