Master-Slave Synchronization of Pulse-Coupled Chaotic Spiking Oscillators

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Abstract - This paper studies basic dynamics of a chaotic spiking oscillator and its master-slave type pulse-coupled system. The slaves can exhibit synchronous phenomena for the chaotic spike-trains of the masters not depending on the initial states. Using the mapping procedure, we analyze the synchronous phenomena. Constructing a simple test circuit, we verify typical phenomena in the laboratory.

I. INTRODUCTION

Spiking neuron models are known as simple single neuron models and have been studied intensively. Applying a periodic stimulus, the neuron exhibits interesting periodic and aperiodic phenomena [1][2]. Connecting plural spiking neuron models, a pulse-coupled neural network (ab. PCNN) can be constructed. The PCNN exhibits a variety of synchronous and asynchronous phenomena [3]-[6]. Analysis of such phenomena is very important not only as basic nonlinear problems but also for engineering applications, e.g., artificial neural networks. However, the most spiking neuron models are two dimensional systems having a single continuous state variable and a single firing state variable. They exhibit only a simple periodic phenomenon and generate only a simple equidistant spike-train. On the other hand, it is known that biological neurons have rich dynamics. For instance, a biological neuron called pacemaker neuron can exhibit interesting bifurcation phenomena including chaos [7]. Recently, a novel spiking neuron model called a resonate-and-fire neuron model has been proposed [8]. It can exhibit rich phenomena, which is like the phenomena of the biological neurons.

In this paper, we study basic dynamics of a Chaotic Spiking Oscillator (ab. CSO). It is a three dimensional system having two continuous state variables and a firing state variable. The continuous state variables can vibrate below a threshold. If the states reach the threshold, the CSO generates a single spike and the states are reset to a base level. Repeating this manner, the CSO can exhibit chaos and can generate chaotic non-equidistant spike-train.

Next, we construct a master-slave type pulse-coupled system of the CSOs. In the system, plural masters and plural slaves exist. All the masters are not coupled and are independent. Each slave is coupled to all the masters by the spike-trains which the masters generate. For the mixed chaotic non-equidistant spike-train, the slaves can exhibit unique attractors not depending on the initial states. It means that the slaves synchronize with the irregularity spike-train. In order to analyze such synchronous phenomena, we derive a return map. Using this map, we discuss the stability of the synchronous phenomena. Since the system is piecewise linear, the return map is given analytically. Also, we analyze the quantitative characteristic of each unit in the system calculating spike rate. Such analysis is fundamental for understanding the dynamics of various PCNN. We have preliminary results of basic pulse-coupled CSOs [9][10]. However, detail analysis has not been sufficient so far.

Finally, we construct a simple test circuit of the CSO as shown in Fig. 1. The unit CSO can be constructed by using two capacitors, two operational transconductance amplifiers (OTAs), a comparator (COMP), a pulse generator (PG), and a firing switch. Using the test circuit, we verify typical phenomena in the laboratory.

II. CHAOTIC SPIKING OSCILLATOR (CSO)

The CSO in Fig. 1 will be a unit element of the pulse-coupled system. The OTA pair construct a two-port voltage-controlled current source (2PVCCS). Connecting two capacitors to both ports, the circuit dynamics is described by the following equation if the firing switch S is opened.

\[
\frac{d}{dt} \begin{bmatrix} C_1 \ v_1 \\ C_2 \ v_2 \end{bmatrix} = \begin{bmatrix} 0 & g_1 \\ -g_2 & g_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{for } v_1(t) < V_T. \quad (1)
\]
We assume that the characteristic roots of Equation (1) are unstable complex conjugates \( \delta \omega \pm j\omega \) and \( \delta > 0 \):

\[
\omega^2 = \frac{g_1 g_2}{C_1 C_2} - \left( \frac{g_2}{2C_2} \right)^2 > 0, \quad \delta = \frac{g_2}{2\omega C_2} > 0.
\]

(2)

Then, the capacitor voltages vibrate below the threshold voltage. If the capacitor voltage \( v_1 \) reaches the threshold voltage \( V_T \), the COMP triggers the PG to generate single spike signal \( v_o \). The spike signal closes the firing switch \( S \) and \( v_1 \) is reset to the base voltage \( E \) instantaneously, holding \( v_2 = \text{constant} \):

\[
\begin{bmatrix}
  v_1(t^+) \\
  v_2(t^+)
\end{bmatrix} = \begin{bmatrix}
  E \\
  v_2(t)
\end{bmatrix}, \quad \text{if } v_1(t) = V_T.
\]

(3)

Here, we refer to this switching as self-firing. Repeating this manner, the CSO generates a spike-train,

\[
v_o(t) = \begin{cases} 
  V_H, & \text{if } v(t) = V_T, \\
  V_L, & \text{for } v(t) < V_T,
\end{cases}
\]

(4)

where \( V_H \) and \( V_L \) are the high and low voltage levels of the PG, respectively. Using the following dimensionless variables and parameters:

\[
\begin{align*}
\tau &= \omega t, \quad x = \frac{v_1}{V_T}, \\
y &= \frac{1}{V_T} \left( -\delta v_1 + \frac{g_1}{\omega C_1} v_2 \right), \\
z &= \frac{v_0 - V_L}{V_H - V_L}, \quad q = \frac{E}{V_T},
\end{align*}
\]

Equations (1), (3) and (4) are transformed into Equations (6), (7) and (8), respectively.

\[
\begin{align*}
\frac{d}{d\tau} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \delta & 1 \\ -1 & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{for } x(\tau) < 1, \\
\begin{bmatrix} x(\tau^+) \\ y(\tau^+) \end{bmatrix} &= \begin{bmatrix} q \\ y(\tau) + \delta(1-q) \end{bmatrix}, \quad \text{if } x(\tau) = 1,
\end{align*}
\]

(6)

(7)

where \([x, y]^T\) is normalized state vector and \( z \) is normalized output spike-train. Note that this normalized equation has two parameters: the damping \( \delta \) and the base level \( q \). Below the threshold \( x = 1 \), Equation (6) has the exact piecewise solution:

\[
\begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} = e^{\delta \tau} \begin{bmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix},
\]

(9)

where \([x(0), y(0)]^T\) denotes an initial state vector at \( \tau = 0 \).

In order to derive a 1-D return map, we define

\[
L = \{ (x, y) \mid x = q \}.
\]

(10)

If the trajectory starts from a point \((q, y_0)\) on \( L \) at \( \tau = \tau_0 \), it rotates divergently around the origin, reaches the threshold at finite time \( \tau_1 \) and jumps to a point \((q, y_1)\) on \( L \). Then, we can define the following 1-D return map.

\[
F : L \to L, \quad y_0 \mapsto y_1.
\]

(11)
Using Equations (7) and (9), this map is given exactly.
\[ y_1 = e^{i \hat{h}} (y_0 \cos \theta_0 \sin \theta_0 + q \sin \theta_0) + \delta (1 - q), \]
where \( \theta_0 \equiv \tau_1 - \tau_0 \). \( \hat{h} \) is the positive minimum root of the following implicit equation:
\[ h(y_0, \theta_0) = e^{i \hat{h}} (y_0 \sin \theta_0 + q \cos \theta_0) - 1 = 0. \]
As we let spike interval between \( i \)-th spike time and \((i + 1)\)-th spike time be \( \theta_i(\equiv \tau_{i+1} - \tau_i) \), we define the following spike rate:
\[ \gamma \equiv \frac{1}{\lim_{K \to \infty} \frac{1}{K} \sum_{i=0}^{K-1} \theta_i}. \]

Fig. 3, 4, and 5 show the diagram of the attractor of the return map, the distribution of the spike interval, and the spike rate, respectively. In each figure, the control parameter is base level \( q \). We can see that the CSO exhibits only chaotic attractor and generates chaotic non-equidistant spike-train.

III. MASTER-SLAVE PULSE-COUPLED CSOS

Next, we consider a master-slave type pulse-coupled system of the CSOs as shown in Fig. 6. The system dynamics is described by

Master units:
\[
\frac{d}{d\tau} \begin{bmatrix} x_{An} \\ y_{An} \end{bmatrix} = \begin{bmatrix} \delta_{An} & \omega_{An} \\ -\omega_{An} & \delta_{An} \end{bmatrix} \begin{bmatrix} x_{An} \\ y_{An} \end{bmatrix},
\]
for \( x_{An}(\tau) < 1 \),
\[
\begin{cases}
q_{An} & \text{if } x_{An}(\tau) = 1,
q_{An} y_{An}(\tau) + \delta_{An} (1 - q_{An}) & \text{if } x_{An}(\tau) = 1,
1 & \text{if } x_{An}(\tau) = 1,
0 & \text{if } x_{An}(\tau) < 1,
z_{An}(\tau) = \begin{cases} 1, & n \in \{1, 2, \ldots, N\} \\
0, & n \in \{1, 2, \ldots, N\} \end{cases}
\end{cases}
\]
Slave units:
\[
\frac{d}{d\tau} \begin{bmatrix} x_{Bm} \\ y_{Bm} \end{bmatrix} = \begin{bmatrix} \delta_{Bm} & \omega_{Bm} \\ -\omega_{Bm} & \delta_{Bm} \end{bmatrix} \begin{bmatrix} x_{Bm} \\ y_{Bm} \end{bmatrix},
\]
for \( x_{Bm}(\tau) < 1 \) and \( x_{Bm}(\tau) < 1 \),
\[
\begin{cases}
q_{Bm} & \text{if } x_{Bm}(\tau) = 1,
q_{Bm} y_{Bm}(\tau) + \delta_{Bm} (1 - q_{Bm}) & \text{if } x_{Bm}(\tau) = 1,
1 & \text{if } x_{Bm}(\tau) = 1,
0 & \text{if } x_{Bm}(\tau) < 1,
z_{Bm}(\tau) = \begin{cases} 1, & n \in \{1, 2, \ldots, N\} \\
0, & n \in \{1, 2, \ldots, N\} \end{cases}
\end{cases}
\]

Since each master is not coupled to the other masters, each master behaves as an independent single CSO. Therefore, any masters can not synchronize with the other masters. Each slave is coupled to all the masters by the spike-trains which the masters generate. If one of the masters, Unit \( An \), reaches the threshold, it exhibits self-firing and is reset to \( q_{An} \), instantaneously. At the same time, all the slaves are reset to \( p_{Bm} \), respectively. We refer to this switchings of the slaves as compulsory-firing. On the other hand, if one of the slaves, Unit \( Bm \), reaches the threshold, only Unit \( Bm \) exhibits self-firing and is reset to \( q_{Bm} \), instantaneously. Simultaneous occurrence of self-firing and compulsory-firing is measure zero for initial states. Formally we define that the self-firing is prioritized in this particular case. Note that all the masters exhibit self-firing only, and have a single base level \( q_{An} \). Also, note that all the slaves have two kinds of firing dynamics, self-firing and compulsory-firing, and have two kinds of base levels \( q_{Bm} \) (for self-firing) and \( p_{Bm} \) (for compulsory-firing). Hereafter, for simplicity of the analysis, we set the parameters as the following:
\[
\delta_{An} = \delta_{Bm} = \delta, \quad \omega_{An} = 1, \quad \omega_{Bm} = \omega_B, \quad q_{An} = q_{Bm} = q, \quad p_{Bm} = p,
\]
that is, this system has four parameters, the damping \( \delta \), the common self-firing base \( q \), the common compulsory-firing base \( p \), and the frequency of the slaves \( \omega_B \).

Figs. 7 and 8 show the typical phenomena in 2-master and 2-slave pulse-coupled system (\( N = 2, M = 2 \)). As shown in Fig. 7, the masters exhibit chaotic attractor and do not synchronize to each other. To both slaves, the mixed spike-train \( z_{A1} + z_{A2} \) which the masters generate is applied. In Figs. 8(a) and (b), the slaves exhibit synchronous phenomena and do not exhibit self-firing. It means that the slaves are synchronize with the mixed spike-train \( z_{A1} + z_{A2} \). In Figs. 8(c) and (d),
Fig. 7. Typical phenomena of the masters ($\delta = 0.07, q = -0.3$).

Fig. 8. Typical phenomena of the slaves ($\delta = 0.07, q = -0.3$). The masters of these slaves are shown in Fig. 7.

The slaves can reach the threshold and can exhibit self-firing. Since the slaves do not couple to each other, the self-firing timing of each slave is different. Therefore, the slaves exhibit temporal synchronous phenomena. As $\theta_{il}$ denotes spike interval between $i$-th spike time and $(i + 1)$-th spike time of Unit $l$, we define the spike rate as the following equation:

$$\gamma_l \equiv \lim_{K \to \infty} \frac{1}{K} \sum_{i=0}^{K-1} \theta_{il},$$

where $l \in \{A1, A2, B1, B2\}$. Fig. 9 shows the spike rate of $z_{B1}$. In the black region, the spike rate is zero and the slaves
can exhibit synchronous phenomena. Hereafter, we focus on this region. In order to analyze the synchronous phenomena, we define a return map. Since the slaves can not reach the thresholds in the black region, we remove the thresholds of the slaves for simplicity of the analysis. First, we define

\[ L_{A1} = \{ x \mid x_{A1} = q \}, \]
\[ L_{A2} = \{ x \mid x_{A2} = q \}, \]
\[ L_A = \{ x \mid L_{A1} \cup L_{A2} \}, \]
\[ \mathbf{x} \in \{ x_{A1}, x_{A2}, x_{A1}, x_{A2}, x_{B1}, x_{B1}, x_{B2}, x_{B2} \}. \]

When the trajectory starts from \( x_0 \) on \( L_A \) at \( \tau = \tau_0 \), each unit rotates divergently around each origin, independently. At the finite time \( \tau = \tau_1 \), either master must reach the threshold. Then it exhibits self-firing and is reset to self-firing base \( q \). At the same time, both slaves exhibit compulsory-firing and are reset to compulsory-firing base \( p \), respectively. In this moment, the state of another master holds constant. Let the point on \( L_A \) be \( x_1 \). Then we can define the following return map.

\[ F : L_A \rightarrow L_A, \quad x_0 \mapsto x_1. \]  

(20)

Using this map, we can calculate the error expansion ratio between the slaves.

\[ \frac{\Delta y_{B1}}{\Delta y_{B0}} = \frac{y_{B1} - y_{B2}}{y_{B1} - y_{B2}}. \]  

(21)

This ratio can be calculated using the following exact solutions.

\[ \frac{\Delta y_{B1}}{\Delta y_{B0}} = e^{\omega q \theta_{A0}} (\cos \omega q \theta_{A0} + \delta \sin \omega q \theta_{A0}). \]  

(22)

where \( \theta_{A0} \equiv \tau_1 - \tau_0 \), \( \theta_{A0} \) is given by the following equation:

\[ \theta_{A0} = \min(\theta_{A1,0}, \theta_{A2,0}). \]  

(23)

Using Equation (20) and (21), we can calculate the average error expansion ratio between the slaves.

\[ \alpha = \lim_{K \to \infty} \frac{1}{\sum_{i=0}^{K-1}} \ln \left| \frac{\Delta y_{B,i+1}}{\Delta y_{B,i}} \right|. \]  

(26)

If \( \alpha > 0 \), the error between the slaves increases and each slave does not synchronize to each other. If \( \alpha < 0 \), each slave can synchronize to each other and the synchronous state is stable not depending on the initial states. Also, it means that the slaves synchronize with the mixed spike-train \( z_{A1} + z_{A2} \). In the figures, if \( q < q_s \), the master trajectories diverge and their attractors do not exist. If \( q_s \leq q < 1 \), both master attractors
exist. In this case, the slave attractors also exist. Since \( \alpha \) is always negative for \( q_r \leq q < 1 \), the synchronous state is always stable.

IV. LABORATORY EXPERIMENTS

We implement a test circuit of the CSO as shown in Fig. 1. In the circuit, the OTAs are realized by using NJM13600D. The COMP, PG, and switch S are realized by using LM339, 4538, and 4066, respectively. Figs. 13 and 14 show the typical synchronous phenomena in the laboratory experiments. In Fig. 14(a) and (b), the slaves do not reach the thresholds and they exhibit synchronous phenomena. In Fig. 14(c) and (d), the slaves can reach the thresholds and they exhibit temporal synchronous phenomena.

V. CONCLUSION

We have studied basic dynamics of the master-slave type pulse-coupled system of the CSOs. We have shown that the slaves synchronize to the mixed spike-train of the masters. Using a return map, we have analyzed the stability of the synchronous state. Also, we have analyzed the quantitative characteristic of the system calculating spike rate. Using a simple test circuit, we have verified typical synchronous phenomena in the laboratory. In order to develop this study, we are considering (1) detail analysis of bifurcation phenomena of the networks, (2) extension to large scale PCNN systems.

References