Synchronization via Multiplex Pulse Trains

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Abstract — The master–slave synchronization of a complex chaotic system using multiplex pulse trains is considered. Each master outputs a chaotic pulse train of narrow pulses, the intervals of which are governed by a chaotic map. The slave system has the winner-take-all function. A simple realization of the systems is shown. We provide a theorem which guarantees the synchronization in the case that all masters and passive slaves have identical parameter values. Experiments confirm that the synchronized state changes intermittently and we clarify this mechanism. Furthermore, in order to obtain robust synchronization, a stabilization technique of the chaotic system is proposed and applied. The stabilization is guaranteed theoretically and some experimental results are given.

Index Terms — Chaos, demultiplexing, multiplex communication, multiplexing, pulse train, synchronization, winner-take-all.

I. INTRODUCTION

RECENTLY, synchronization phenomena in chaotic systems have been treated intensively, e.g., chaos synchronization in master–slave systems and mutually coupled systems [1]–[10]. There are many studies on the application of such phenomena to communication systems (see [11] and references therein). In these systems the chaotic signals are treated as continuous- or discrete-time analog valued signals. Then the synchronized state can sometimes be broken by disturbances and/or distortions as discussed in [12].

We consider the chaos synchronization using multiplex pulse trains. In our system, the information for synchronization is contained in the time intervals and not in the signal value. Synchronization by such a pulse train is an interesting topic since the pulse trains can be transmitted via a noisy channel. Synthesis and analysis of such chaotic pulse-train generators (CPG’s) has been studied recently [9], [10], [13]–[19]. Also, the synchronization of multiple chaos by means of a multiplex signal is interesting from the theoretical viewpoint of chaos synchronization and might be applicable in multiplex communication schemes with chaos, as discussed in [20]–[22]. The objectives of this paper are the following.

• Design of the System Structure: A master–slave system scheme is considered. In the master system, the multiplex signal is obtained by simply adding the pulse trains. The slave system has the winner-take-all function.

• Analysis of the Synchronization Behavior: We prove a theorem which guarantees the synchronization between the master system and passive slave system where all masters and slaves have identical parameter values. We also analyze an implemented circuit.

• Improvement of the Synchronization Performance: Stabilized periodic pulse trains are used in order to obtain robust synchronization.

• Personalization of the Master–Slave Pairs: In order to personalize the masters and the slaves, different parameters are assigned to different master–slave pairs.

Experiments confirm the personalization.

The master is a CPG [13], [14] which is a simplified version of a forced relaxation oscillator discussed in [23] and [24]. The pulse trains are generated by one-dimensional (1-D) return maps which can easily be changed, thus providing the generation of a variety of pulse trains.

Experiments confirm that chaos synchronization is intermittent. Intermittent synchronization has been studied recently [6]–[8] and we clarify this phenomenon for our structure.

In order to obtain robust synchronization, a stabilization using a higher frequency external periodic signal is proposed and applied. This leads to a discretization of the return map [13], [14], and the resulting finite state system generates periodic signals only. Although some experimental results on taming chaos by periodic perturbations have been reported [10]–[26], we treat this from a more theoretical viewpoint. (Some comments on the words taming and stabilizing are given in Section III.) Finally, using a stabilization procedure, we confirm analytically and experimentally that the system shows robust synchronization.

II. SYNCHRONIZATION VIA MULTIPLEX PULSE TRAINS

A. The System

1) General Principle: We consider the synchronization behavior in the master–slave system depicted in Fig. 1. In the master system, there are \(N\) masters which have the discrete-time states \(\theta_i(n), n = 1, 2, \ldots\). The masters output pulse trains \(y_i(t)\). These pulse trains are multiplexed by simply adding and the multiplex signal \(p(t)\) is sent to the slave system via a single line. In the slave system, there are \(N\) slaves and the pulse distributor, which distributes incoming pulses to the slaves. The pulse distributor has the winner-take-all function.

The \(n\)th master outputs the pulse train \(y_n\), as shown in Fig. 1(b), where the \(n\)th pulse position is denoted by \(\tau_n(n)\) and the pulse interval between the \(n\)th and \(n+1\)st pulse positions is denoted by \(\Delta \tau_n(n) = \tau_n(n+1) - \tau_n(n)\). The pulse intervals

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The master–slave system. (a) Whole scheme. (b) The output pulse trains $y_i$ of the masters and the multiplex signal $p(\tau)$.

From the $i$th master are given by

$$\Delta \tau_i(n) = \phi_i(\theta_i(n); p_i), \quad \theta_i(n) \in [0, 1)$$

where the $\phi$ is a linear scaling function and $p_i = (r_i, s_i)$ is the parameter vector of the $i$th master and slave. The state $\theta_i(n)$ is governed by the following return map:

$$\theta_i(n+1) = f(\theta_i(n); p_i) = m(r_i \theta_i(n) + s_i)$$

for $n = 1, 2, \cdots$, $m(\theta) = \theta(\text{mod} 11)$

$$f_i(0, 1) \rightarrow (0, 1), \quad r_i > 1, \quad s_i > 0.$$  \hspace{1cm} (1)

An example of the return map $f$ is shown in Fig. 2. According to [27] and [28], the system (1) will generate chaos for $r_i > 1$: the masters generate chaotic pulse trains $y_i(\tau)$.

We consider two kinds of slave structures. One is the same structure as the master and can oscillate autonomously and the other one is a subsystem of the master, which cannot oscillate by itself. The states of the slaves $\theta_i^s$ are fed back to the pulse distributor, which is designed to realize the synchronization.

2) Master Structure: In order to realize the master, the structure in Fig. 3 could be used [15], [16]. In this figure, $D$ is a controlled analog delay. The value-time converter outputs successive instantaneous pulses. The intervals between them are $\Delta \tau_i(n)$. At every pulse moment, the time-value converter clocks the analog delay $D$ and gets the next pulse interval $\Delta \tau_i(n+1)$.

Here we introduce a system structure (see Fig. 4) which directly generates the pulse train without using a time-value converter. This structure is a simplified version of a forced relaxation oscillator discussed in [23] and [24]. The basic unit (BU) consists of an integrator, an adder, and the controlled switches. The pulse generator (PG) consists of the BU, the comparator, and the pulse former. The masters’ state equations are given by

$$\frac{dx_i(\tau)}{d\tau} = \frac{1}{\lambda_i}, \quad \text{for } x_i(\tau - 0) < 0$$

$$x_i(\tau + 0) \text{ is set to } d_i(\tau), \quad \text{if } x_i(\tau - 0) = 0$$

$$y_i(\tau) = \begin{cases} 1, & \text{if } x_i(\tau - 0) = 0 \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (2)

where $\tau$ is the normalized time, $x_i$ is the continuous-time state of the $i$th master, $N$ is the number of the masters and the slaves, and $\lambda_i > 0$ is a system parameter. The signal $d_i(\tau)$ is an internal signal of the $i$th BU which is given by

$$d_i(\tau) = b(\tau) - a_i < 0$$  \hspace{1cm} (3)

where $a_i$ is the second system parameter and $b(\tau)$ is periodic with period one and is called the base signal, defined as

$$b(\tau) = -m(\tau).$$  \hspace{1cm} (4)

The time-domain waveforms are shown in Fig. 5. The state of the $i$th master starting from $x_i(0)$ increases and reaches
the threshold value \( x_i = 0 \). At this moment the comparator switches, an instantaneous pulse is generated by the pulse former, and the state is set to \( d_i(\tau) \). After this, the process is repeated. Recall that the \( q \)th pulse position of the \( q \)th master is denoted by \( \tau_q(n) \). Then the \( q \)th master outputs a pulse train such that

\[
\theta_q(n+1) = \begin{cases} 
1, & \text{if } \tau = \tau_q(n), \\
0, & \text{otherwise},
\end{cases} \quad n = 1, 2, \ldots
\]

Here we analyze the pulse train by means of the pulse intervals, in which the information for synchronization is contained. The pulse interval \( \Delta \tau_q(n) \) is determined by the value of the internal signal \( d_q(\tau_q(n)) \) which is periodic with period one. Hence, for convenience, we introduce the restricted time coordinate \( \theta_q \), as shown in Fig. 5. Then the pulse positions \( \theta_q(n) \) are governed by the following pulse position return map:

\[
\theta_q(n+1) = m(\theta_q(n) + \Delta \tau_q(n)) = m((1 + \lambda_i)\theta_q(n) + \alpha_s \lambda_i)
\]

which is identical to the return map (1) by transforming the parameters as

\[
r_i = 1 + \lambda_i, \quad s_i = \lambda_i \alpha_i.
\]

The pulse interval \( \Delta \tau_q(n) \) is given by (see Fig. 5)

\[
\Delta \tau_q(n) = \phi(\theta_q(n); \lambda_i, \alpha_i) = \lambda_i \theta_q(n) + \alpha_i
\]

\[
\phi : [0, 1) \rightarrow \{ \lambda_i \alpha_i, \lambda_i + \alpha_i \}
\]

\[
\Delta \tau_q(n) \in [\lambda_i \alpha_i, \lambda_i + \alpha_i]
\]

\[
\theta_q(1) = m((-\lambda_i \theta_q(0))).
\]

Using the pulse position return map \( f \) and the scaling \( \phi \), we get the return map for the pulse intervals \( \Delta \tau_q(n) \) as

\[
\Delta \tau_q(n + 1) = F(\Delta \tau_q(n))
\]

\[
= \phi \circ f \circ \phi^{-1}(\Delta \tau_q(n))
\]

\[
= m \left( \frac{1}{\lambda_i}(1 + \lambda_i)\Delta \tau_q(n) - \lambda_i \alpha_i \right) + \lambda_i \alpha_i
\]

\[
F : \{ \lambda_i \alpha_i, \lambda_i + \alpha_i \} \rightarrow \{ \lambda_i \alpha_i, \lambda_i + \alpha_i \}.
\]

Note that the pulse interval \( \Delta \tau_q(n) \) and the pulse position \( \theta_q(n) \) are equivalent via the scaling \( \phi \), hence, we analyze the pulse train by using \( \theta_q(n) \).

The absolute pulse positions \( \tau_q(n) \) are given by the difference equation

\[
\tau_q(n+1) = \tau_q(n) + \Delta \tau_q(n), \quad n = 1, 2, \ldots
\]

with the solution

\[
\tau_q(n) = \tau_q(1) + \sum_{i=1}^{n-1} \Delta \tau_q(i), \quad n = 2, 3, \ldots
\]

Since \( f \) generates chaos, the masters in (2) generate chaotic pulse trains \( y_k(\tau) \). Hence, we call each master a CPG. The graph of the return map \( f \) is determined by the form of the base signal \( b(\tau) \). Hence, by adjusting the base signal, the CPG can produce a variety of pulse trains. Return maps of the form (1) have been treated intensively (e.g., [29]).

Then, using the substructures of the CPG, the master system and the slave system are realized, as shown in Fig. 6. The master system consists of \( N \) PG's (see Fig. 4), a common base signal generator (BSG), and an adder. The multiplex signal \( p(\tau) \) is obtained by simply adding the chaotic pulse trains

\[
p(\tau) = \sum_{i=1}^{N} y_k(\tau), \quad 1 \leq k \leq N.
\]
by

\[
\begin{cases}
    x'_i(\tau + 0) = \text{set to } d'_i(\tau), & \text{if } p(\tau - 0) = 1 \\
    \frac{dx'_i(\tau)}{d\tau} = \frac{1}{\lambda_i}, & \text{if } p(\tau - 0) = 0 \\
    \text{otherwise,} & i = 1, 2, \ldots, N
\end{cases}
\]

(13)

where \( x'_i \) is the state of the \( i \)th BU and \( d'_i(\tau) = b'(\tau) - a_i \) is the internal signal of the \( i \)th BU. According to this algorithm, the input pulse train of the \( i \)th BU will be \( y'_i(\tau) \).

Since this slave system cannot oscillate without the signal \( p(\tau) \), it is called the passive slave system.

Second, we consider a slave system which consists of slave generators [see Fig. 6(b)]. The terminal (b) of the \( i \)th slave generator is connected to the switch \( S_i \). Then the slave system is described by

\[
\begin{cases}
    x'_i(\tau + 0) = \text{set to } d'_i(\tau), & \text{if } p(\tau - 0) = 1 \\
    \frac{dx'_i(\tau)}{d\tau} = \frac{1}{\lambda_i}, & \text{if } p(\tau - 0) = 0 \\
    \text{or } x'_i(\tau - 0) \geq \epsilon, & \text{otherwise}
\end{cases}
\]

(14)

where \( \epsilon > 0 \) is a small value. The input pulse train of the \( i \)th slave generator will be

\[
y'_i(\tau) = \begin{cases}
    1, & \text{if } p(\tau - 0) = 1 \\
    x'_i(\tau - 0) = \max_j \{x'_j(\tau - 0)\} \\
    0, & \text{otherwise}
\end{cases}
\]

Since this system can oscillate by itself, we call it the active slave system. From active slave systems a better synchronization performance can be expected than from passive ones.

4) Implementation: Fig. 7(a) shows an implementation example of the BSG and the CPG. MF is the mono-stable flip flop which outputs an instantaneous pulse if its input changes from low (\(-V\)) to high (\(V\)). In this example, the current sources are
(a) Implementation example. (b) Measured time-domain waveforms.

Fig. 7(b) shows measured waveforms from the implemented circuit.

In Fig. 8(a) an implementation example of the master system and the slave system is depicted. In the master system, the multiplexing is realized by the logical OR gate which outputs the multiplex signal \( P(t) \). In the slave system, the base signal reconstructor (BSR) [see Fig. 8(b)] produces \( B'(t) \) which repeats \( B(t) \) using the periodic clock pulse train \( C(t) \). The slave generator is shown in Fig. 8(c). Using the transformation (16) and \( \epsilon = V_c/V_B \), its circuit equation is transformed to the slave generator in (14). The passive slave system (13) is realized using the BU’s, and the active slave system (14) is realized using the slave generators, as described in point 3).

The dynamic winner-take-all circuit consists of the maximum value selector, the comparators, and the sample and hold circuits (S&H’s). The S&H’s sample the slaves’ states \( v_i \) at every incoming pulse moment and hold them until the end of each pulse. All outputs of the OP-amps are compared with \( -V_B > -V \). If \( v_i \) is the maximum, the \( i \)th AND gate is opened and thus the \( i \)th BU (or slave generator) gets the actual pulse for synchronization. In this way, the AND gates realize the switches \( S_i \). By means of the S&H’s, the maximum value selector is decoupled from the BU’s (or slave generators) and racing is avoided.

Note that we assumed that \( \frac{d}{dt} x^2 = 0 \). In the implementation circuit, this can be achieved by the clock pulse train \( C(t) \) which is sent to the slave system via an additional line [see Fig. 8(a)]. A discussion on the clock pulse train is given in Section IV.

B. Behavior Analysis

1) Passive Slave with Identical Parameter Values: First, we give a theoretical result on synchronization between a master system (2) and a passive slave system (13) (see Fig. 6), where all slaves and masters have identical parameters, i.e., \( a_i = a_j \) and \( \lambda_i = \lambda_j \) for all \( i, j \). Thus, the internal signals in both systems are identical, i.e., \( d_i(\tau) = d_j(\tau) = d_j(\tau) \) for all \( i, j \). We assume that the multiplexed pulse trains are disjoint, i.e.,

\[
y_i(\tau) \cdot y_j(\tau) = 0, \quad \text{for } \tau \geq 0, \ i \neq j; \ i, j = 1, 2, \ldots, N.
\]
A necessary condition for that is that the initial states in the master system are disjoint: \( x_i(0) \neq x_j(0), \ i \neq j, \ i, j = 1, 2, \ldots, N \).

Further, we assume that the initial states of the BU’s satisfy
\[
G_i(0) > -\alpha_i, \quad x_i(0) \neq x_j(0), \quad i \neq j, \ i, j = 1, 2, \ldots, N.
\] (18)

That is, each BU has different initial state \( x_i(0) \) which is larger than \(-\alpha_i\). In order to analyze the behavior of the system under the above assumptions, we give some definitions.

**Definition 1:** Suppose that the state \( x_i(0) \) of the \( i \)th BU is set to the internal signal \( d_i^j(\tau_a) \), where \( p(\tau_a) = y_j(\tau_a) = 1 \). Then the \( j \)th BU is said to accept the pulse train \( y_j \) at \( \tau = \tau_a \). The
Fig. 9. Time-domain waveforms from the master system and the passive slave system.

The uth BU is said to trace $x_j$ for $\tau_a < \tau < \tau_b$ if $x_j^u = x_j$ for $\tau_a < \tau < \tau_b$.

Suppose that the uth BU traces $x_j$ for $\tau_a < \tau < \tau_b$ and the jth BU accepts $y_j$ at $\tau = \tau_d$. Then the uth BU is said to be intercepted by the jth BU at $\tau = \tau_d$.

Fig. 9 shows time-domain waveforms from both the master system and the slave system for $N = 2$. At $\tau = \tau_1(1)$ the second BU accepts the pulse train $y_1$ and traces $x_1$ for $\tau_1(1) < \tau < \tau_1(2)$. At $\tau = \tau_1(2)$ the first BU accepts $y_1$ although the second BU traced $x_1$. Then the second BU is intercepted by the first BU. The second BU accepts $y_2$ at $\tau = \tau_2(1)$ and traces $x_2$ for $\tau > \tau_2(1)$. Also, the first BU traces $x_1$ for $\tau > \tau_1(2)$.

Definition 2: The master system and the slave system are said to be synchronized for $\tau_a < \tau < \tau_b$ if there exist a unique BU which traces $x_i$ for all $i = 1, 2, \cdots, N$ during $\tau_a < \tau < \tau_b$.

In Fig. 9 the synchronization is achieved for $\tau \geq \tau_2(1)$. Note that after $\tau = \tau_2(1)$, the slave states $x_i^u$ are set to $d_i^u(\tau)$ only when they are equal to zero. This property can be used as a criterion for synchronization.

Definition 3: Let $M$ be the maximum (latest) pulse position in $\{\tau_1(1), \tau_2(1), \ldots, \tau_N(1)\}$ and let $\tau_i$ be the maximum pulse position in $\{\tau_1(1), \tau_i(2), \cdots\}$ such that $\tau_i(n) \leq M$.

Fig. 10. Time-domain waveforms from the passive slave system.

In Fig. 9 these are given by $M = \tau_2(1), T_1 = \tau_1(2)$ and $T_2 = \tau_2(1)$.

Definition 4: If a BU traces some pulse train $y_i$ for $\tau_a < \tau < \tau_b$, the BU is said to be a tracing basic unit (TBU) for $\tau_a < \tau < \tau_b$, otherwise said to be a nontracing basic unit (NBU) for $\tau_a < \tau < \tau_b$. If a pulse train is traced by some TBU, the pulse train is said to be a traced pulse train (TPT), otherwise it is said to be a nontraced pulse train (NPT).

In Fig. 9, the second BU is a TBU and correspondingly, $y_2$ is TPT for $\tau_a < \tau < \tau_2(2)$. The first BU is an NBU for $0 < \tau < \tau_2(2)$ and it intercepts $y_2$ and changes to a TBU at $\tau = \tau_1(2)$. Consequently, the second BU becomes an NBU for $\tau_1(2) < \tau < \tau_2(1)$. The pulse train $y_2$ is an NPT during $0 \leq \tau < \tau_2(1)$.

Then we have the main theorem for any $N$.

Theorem: Suppose that the PG’s in the master system and the BU’s in the passive slave system have identical parameters, i.e., $a_i = a_j$ and $\lambda_i = \lambda_j$ for all $i, j$ and (17) and (18) are satisfied. If the uth BU accepts some pulse train $y_i$ at $\tau = T_j$, then it traces $x_j$ for $\tau \geq T_j$. This statement holds for all $i = 1, 2, \cdots, N$ and, thus, the synchronization is achieved between the master system and the passive slave system for $\tau \geq M$.

Proof: We first show that the number of TBU’s and TPT’s are equal, that is, TBU’s and TPT’s correspond uniquely. Assume (for contradiction) that the number of TBU’s is larger than that of the TPT’s and, thus, there exists at least one TPT which is traced by more than one TBU. In this case, from the initial states of those TBU’s, either at least two are identical or at least one is less than $\tau_a$. This contradicts (18). Also, it follows from (17) that the number of the TBU’s must not be less than that of the TPT’s.

Next we show that a BU traces some state $x_i$ for $\tau \geq T_i$ if it accepts $y_i$ at $\tau = T_i$. We focus on the state of the uth BU, which is described by the bold trajectory in Fig. 10. Let us suppose that the uth BU accepts $y_i$ at $\tau = \tau_i(1) = T_i$. Now for contradiction assume that

$$x_i^u(\tau_i(n) - 0) = \max_{\hat{h}} \{x_i^\hat{u}(\tau_j(n) - 0)\}$$

where $n$ is some integer and that according to the algorithm
Fig. 11. Experimental results from the master–slave systems for \( N = 3 \) with identical values of \( I_j (j = 1, 2, 3) \) and \( V_{j'} (j = 1, 2, 3) \). The parameters are the same as Fig. 7(b). (a) States of the PG’s and the multiplex signal in the master system. (b) Synchronization in the passive slave system. (c) Break of synchronization in the same passive slave system as Fig. 11(b). (d) Time-domain waveforms from the active slave system. \( V_i = 0.3 \) [V].

Fig. 12. Time-domain waveforms from the active slave system. The broken lines are the trajectories \((x'_1, x'_2, x'_3)\) from the passive slave system.

(13), the \( \delta \)th BU accepts \( y_j \) at \( \tau = \tau_j (n) \in \{ T_i, M \} \). At this moment, \( y_j \) changes from TPT to NPT and then there must exist at least one NBU since the number of the NPT’s is equal to that of the NBU’s. For such an NBU two cases are possible:

(A) the BU was an NBU during \( 0 \leq \tau < \tau_j (n) \);
(B) the BU changed from a TBU to an NBU at some moment \( \tau = \tau_j (m + 1) < \tau_j (n) \) and was an NBU during \( \tau_j (m + 1) < \tau < \tau_j (n) \) where \( m \) is some integer.

An example of the trajectories for the two cases are shown in Fig. 10. The NBU (A) satisfies \( x'_j (\tau_j (n)) > x'_j (\tau (n)) \) since \( x'_j (\tau (1)) > x'_j (\tau_j (1)) \) and it does not decrease until \( \tau = \tau_j (n) \). The NBU (B) satisfies \( x'_j (\tau_j (n)) > 0 \) since \( x'_j (\tau_j (m + 1)) = 0 \) and it does not decrease until \( \tau = \tau_j (n) \). Obviously, both cases contradict (19). Q.E.D.

According to this theorem, the synchronization between the whole master system and the whole passive slave system is achieved after a transition period \( M \) if noise does not exist. Note that, which specific slave is synchronized with which master depends upon the initial conditions.

Fig. 11 shows measured time-domain waveforms from an implemented circuit. In Fig. 11(a) the states in the master system and the multiplexed pulse trains are shown. Fig. 11(b) shows waveforms from the passive slave system. It can be seen that the slave system is synchronized to the master system. Fig. 11(c) also shows the snapshot at another moment where synchronization is broken. As shown in these figures, the passive slave system is synchronized intermittently. The reasons for this imperfect synchronization include the following.

R1) The pulses in \( p(\tau) \) may overlap or may be located very close to each other because of the mixing property of chaos [27]. Then the dynamic winner-take-all circuit cannot distinguish these pulses and the synchronization is broken. This will be discussed in the next point 2).

R2) One pulse may be located very close to \( \tau = m = 1, 2, 3, \cdots \) because of the ergodic property of chaos [27]. Then \( x'_j \) may be set to \( d_k (m + 0) = -a_i \) while its master state (synchronized state) is set to \( d_k (\tau - 0) = -1 - a_i \) and vice versa. This also breaks synchronization. We have analyzed this phenomenon in [8].

Referring to [8], [27], and [28] in the return map \( I \), the pulse positions \( \theta_k (n) \) must be located very close to at least one discontinuous point. Since such pulse positions are mapped to very close to \( \theta_k (n + 1) = 0 \) or 1, synchronization is broken.

2) Active Slave System with Identical Values of \( \lambda_i \) and \( a_j \): Next, we consider the active slave system (14). The time-domain waveforms from the master system and the active slave system are shown in Fig. 12. In this figure, the synchronization is achieved at \( \tau = \tau_3 (1) \). Note that the pulses are very close to each other at \( \tau = \tau_3 (3) \approx \tau_3 (2) \). If this occurs in the passive slave system, \( x'_i \) may overshoot zero, the pulse train \( y_2 \) may be intercepted by the first BU, and the synchronized state may be changed (the broken lines in Fig. 12). In the active slave system, \( x'_i \) is set to \( d_k (\tau) \) due to the threshold \( \epsilon > 0 \). Hence, the synchronized state can be held as long as \( x'_i \) is the maximum at \( \tau = \tau_3 (4) \). Fig. 11(c) shows the experimental results for the active slave system for \( N = 3 \): the system exhibits synchronization recovery performance.

3) Different Parameter Values: In order to personalize the slaves and the masters, different parameter values are assigned to different master–slave pairs, i.e., \( \lambda_i, a_i \neq \lambda_j, a_j \) for \( i \neq j \). Fig. 13(a) and (b) shows the experimental results for the passive and active slave systems for \( N = 3 \), where
III. SYNCHRONIZATION VIA MULTIPLEX STABILIZED PULSE TRAINS

A. Stabilization of Chaos by an External Periodic Signal

Here we use a stabilization procedure for the chaotic system (2) by an external periodic signal. The principle of the stabilization technique is first explained by means of the return map.

1) Principle: In order to stabilize the periodic orbits of the return map $f$ in (1), we discretize the state of $f$ and obtain a new return map $g$ (see Fig. 14)

$$\theta(n+1) = g(\theta(n)) = f(\theta(n)); r, s - h(\theta(n))$$

where $h(\theta)$ is given by

$$h(\theta) = \alpha \beta \left( m \left( \frac{\theta - \gamma}{\beta} \right) - \frac{1}{2} \right)$$

$$\alpha = r, 0 < \beta < 1, 0 \leq \gamma < \beta$$

Fig. 14. The principle of stabilization: (a) An example of $f$, unstable periodic points with period three, and corresponding orbits. $r = 2, s = 1$. (b) An example of $h$. $\alpha = 2, \beta = 1/7, \gamma = \beta/2$. (c) The return map $g$.

to $\lambda_d$ are assigned. In Fig. 13(d), the synchronized state is broken at $t = t_e$ due to $R_1$, but is recovered immediately since the active slave has its own threshold (see Fig. 12). The personalization is observed intermittently. In both cases, the active system exhibits a more robust synchronization than the passive one. More detailed analysis will be done in the near future.

Fig. 13. Experimental results from the master–slave systems with different parameter values for $N = 3$. The left snapshots are the time-domain waveforms and the right ones are the planes. (a) Intermittent synchronization from the passive slave system ($V_1 = 2.0$ [V], $V_{A2} = 1.2$ [V], $V_{A3} = 2.6$ [V]). The other parameters are the same as in Fig. 7(b). (b) Intermittent synchronization from the active slave system ($V_1 = 0.3$ [V]). The other parameters are the same as in Fig. 13(a). (c) Intermittent synchronization from the passive slave system ($I_1 = 11$ [mA], $I_2 = 17$ [mA], $I_3 \approx 7.0$ [mA]). The other parameters are the same as in Fig. 7(b). (d) Intermittent synchronization from the active slave system ($V_1 = 0.3$ [V]). The other parameters are the same as in (c).
Fig. 15. Time-domain waveforms from the CPG with the external periodic signal $c_i(\tau)$.

[see Fig. 14(b)]. The slope of $g$ is now zero almost everywhere and the variable $\theta$ is restricted to discrete values. Then all orbits generated by $g$ are periodic and stable. However, it is complicated to determine the periodic solution of (20) for arbitrary parameter values.

2) Realization: The above principle is realized by adding a periodic sawtooth wave (external signal) to the internal signal $x_i$ of the $i$th master in (2). Here $k_i$, $\beta$, and $\gamma$ control the amplitude, frequency, and the phase of the signal $c_i(\tau)$, respectively. An example of $c_i(\tau)$ is shown in Fig. 15. Then the internal signal of the $i$th master is given by

$$d_i(\tau) = b(\tau) - a_i + c_i(\tau)$$

where $b(\tau)$ is the base signal and $a_i$ is an offset. In order to get the stabilized periodic pulse trains, we set

$$k_i = 1 + \frac{1}{N_i}, \quad \frac{1}{\beta} = 1, 2, \ldots$$

The internal signal $d_i(\tau)$ must always be negative in order to guarantee oscillation, hence, $a_i$ must satisfy $b(\tau) - a_i + c_i(\tau) < 0$ for all $\tau$. As a result, instead of the chaotic map (6), we get the new return map

$$\theta_i(n+1) = m(\theta_i(n) - a_i) + \lambda_i \theta_i(n) + \lambda_i c_i(\theta_i(n))$$

which is identical to the system (20) via the transformation (7). Since the system (20) can generate only periodic orbits, the masters can output only periodic pulse trains.

Some experimental results on taming chaos by periodic perturbations have been reported [10]–[26]. The taming of chaos implies changing a chaotic orbit to a periodic one by changing the stability of a chaotic system locally. On the other hand, the stabilization of chaos implies achieving a desired stabilized periodic orbit by some procedure. In this paper, the CPG is tamed by the external signal and we give some theoretical results of the basin of attraction for the stabilized periodic orbits. Then, by setting the initial state appropriately, we can get any desired stabilized periodic orbit. Hence, we use the term stabilization in this paper rather than taming.

3) Example: We consider the important case of $f$ in (1) with

$$r_i = 2, \quad s_i = 1$$

Then the return map $f$ is a Bernoulli map. We cite some basic definitions and results for periodic points and periodic orbits (see, for example, [31] and [32]).

- A point $\theta^p$ is said to be a periodic point of $f$ with period $Q$ if $f^Q(\theta^p) = \theta^p$ where $f^Q$ denotes the $Q$-fold composition of $f$. A sequence $(\theta^p, f(\theta^p), \ldots, f^{Q-1}(\theta^p), \theta^p, \ldots)$ is said to be a periodic orbit. A periodic orbit is said to be stable if $|Df^Q(\theta^p)| \leq 1$ where $Df = df/d\theta$ and unstable otherwise.
- Since $Df = 2$ for all $\theta_i(\eta)$ except for the discontinuous points, all periodic orbits generated by the system (1) are unstable. An example of $f$ unstable periodic points and corresponding unstable periodic orbits is shown in Fig. 14(a).
- All unstable periodic points with period $Q$ are given by

$$\theta^p = \frac{q}{2^Q - 1}, \quad q = 0, 1, \ldots, 2^Q - 2$$

In the return map $f$, an unstable periodic point $\theta^p$ determines an unstable periodic orbit. Therefore, a pulse train, generated by the CPG, started from $\theta^p$, is to be periodic and we refer to it as an unstable periodic pulse train (UPP). A UPP started from $\theta^p$ is denoted by $P(\tau, \theta^p)$. From simple algebra it follows that all UPP’s with period $Q$, started from different periodic points, are disjoint (have no overlapping), i.e.,

$$P(\tau, \theta^p_k) \cap P(\tau, \theta^p_j) = \emptyset, \quad \text{for } \tau \geq 0, \theta^p_j \neq \theta^p_k;$$

$$j \neq k, \quad j, k = 0, 1, \ldots, 2^Q - 2$$

In order to stabilize the unstable periodic orbits of $f$ with period $Q$, we use the function $h(\theta; \alpha_i, \beta, \gamma)$ in (21) with

$$\alpha_i = 2, \quad \beta = \frac{1}{2^Q - 1}, \quad \gamma = \frac{\beta}{2}$$

Then, according to the principle from point 1), the system is governed by the return map

$$\theta_i(n+1) = g(\theta_i(n)), \quad g: [0, 1] \rightarrow \{\theta_0^p, \theta_1^p, \ldots, \theta_{2^Q-2}^p\}$$

Fig. 14(c) shows an example of $g$ for $Q = 3$. Equation (30) implies that all unstable periodic orbits with period $Q$ under the parameter (26) can be stabilized by applying the function $h$ with the parameter values in (29).

Using this result, the external periodic signal $c_i(\tau)$ with the parameters

$$\lambda_i = 1, \quad a_i = 1, \quad \beta = \frac{1}{2^Q - 1}, \quad \gamma = \frac{\beta}{2}$$

implies that all unstable periodic orbits with period $Q$ under the parameter (26) can be stabilized by applying the function $h$ with the parameter values in (29).
is added to the internal signal $d_k(\tau)$ in order to obtain stabilized periodic pulse trains. Under these parameter values, the pulse positions are governed by the return map (30). Therefore, all UPP's with period $Q$ embedded in the chaotic pulse trains can be stabilized, with the exception of $P_k(\tau, \theta_0^k)$ because $\theta_0^k$ is located at the edge point.

The initial state $x_k(0)$ determines the first pulse position $\theta_k(1) = m(\frac{-\lambda}{2} x_k(0))$. Hence, an initial state $x_k(0) \in w_i^k$, $q = 0, 1, \ldots, 2Q - 2$ gives a stabilized UPP

$$y_k(\tau) = P_k(\tau, \theta_0^k)$$

where

$$w_i^k = \left[-\frac{\theta_0^k - \beta}{2}, -\theta_0^k + \frac{\beta}{2}\right].$$

The intervals $w_i^k$ of initial states and the stabilized UPP's have a one-to-one correspondence since the UPP's in (28) are disjoint.

4) Implementation: Fig. 16(a) shows an implementation example of the external periodic signal generator (EPG). The circuit is similar to the BSG in Fig. 7(a). The mono-stable flip-flop MF2 outputs a periodic pulse train $CE(t)$ which synchronizes the BCG. The phase difference can be adjusted by $V_G \in (-V_D, V_D]$. Then the base signal $B(t)$ and the external signal for the $i$th master $C_i(t)$ are given by

$$B(t) = -K m \left(\frac{t}{T}\right), \quad K = \frac{I_B}{C_B}, \quad C_i(t) = k_i C_0(t)$$

$$C_0(t) = K T_\beta \left(m \left(\frac{t - G}{T_\beta}\right) - \frac{1}{2}\right), \quad T_\beta = \frac{2V_D}{K},$$

$$G = \frac{V_D - V_G}{K}$$

with the function $m$ defined in (1) where $T_\beta$ is the period of the external signal $C_i(t)$ and the period of the base signal is given by $T = T_\beta \text{Int}(V_B/(2V_D) + 1)$. The external periodic signal is applied to the PG's (see Fig. 6) in the master system, as shown in Fig. 15(b). Then the master system is described by (2) and (23) with the transformation (16) and

$$\beta = \frac{T_\beta}{T}, \quad \gamma = \frac{G}{T}, \quad c_i(\tau) = \frac{C_i(T_\tau)}{V_B}.$$
Fig. 17. Experimental results from the master–slave systems with the stabilized periodic pulse trains. (a) Time-domain waveforms from the master system (the upper left) and the passive slave system (the lower left) and planes (the right) with identical parameter values. The parameters are the same as in Figs. 11 and 16. (b) Time-domain waveforms from the master system (the upper left) and the passive slave system (the lower left) and planes (the right) with different parameter values. The parameters are the same as in Figs. 13(a) and 16.

Fig. 18. Generalized scheme of the stabilization.

TABLE I

<table>
<thead>
<tr>
<th>$S_b$</th>
<th>$S_c$</th>
<th>$d(\tau)$</th>
<th>Pulse-train $y(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>open</td>
<td>open</td>
<td>$-a$</td>
<td>periodic</td>
</tr>
<tr>
<td>closed</td>
<td>open</td>
<td>$-a + b(\tau)$</td>
<td>chaotic</td>
</tr>
<tr>
<td>closed</td>
<td>closed</td>
<td>$-a + b(\tau) + c(\tau)$</td>
<td>stabilized periodic</td>
</tr>
</tbody>
</table>

A generalized scheme of the stabilization procedure considered in this paper is depicted in Fig. 18. The periodic signal generator 1 (corresponding to PG in Fig. 4) has an output $y(\tau)$ and an input $d(\tau)$ which is determined by the switches $S_b$ and $S_c$. The periodic signal generator 2 (corresponding to the BSG in Fig. 4) outputs $b(\tau)$, and the periodic signal generator 3 outputs a higher frequency signal $c(\tau)$ [corresponding to the external periodic signal in (22)] than $b(\tau)$. Here, $b(\tau)$ and $c(\tau)$ are synchronized. In dependence from the switches the following cases, shown in Table I, can be distinguished.

If $S_c$ is open and $S_b$ is closed, i.e., $d(\tau) = -a + b(\tau)$, the pulse train $y(\tau)$ will be periodic or chaotic depending on the parameter values of $b(\tau)$. Note that we treated the case where $b(\tau) = -a + b(\tau)$ [in (4)] and $\lambda_4 > 0$ [in (2)], hence the system generates chaos only.

We clarified theoretically that the frequency of the external periodic signal $c(\tau)$ determines the properties (e.g., period) of the stabilized periodic orbits. The stabilizing chaos technique by external periodic signals is simpler than the ones described in [33] and can be applied up to higher frequencies since it does not require state feedback [26]. Our results contribute to the theoretical analysis of both stabilization and bifurcation phenomena in periodically perturbed chaotic systems [25], [26].

IV. CONCLUSIONS

We have considered the master–slave synchronization via multiplex pulse trains. Two slave structures, a passive one
TABLE II
SYNCHRONIZATION PHENOMENA

<table>
<thead>
<tr>
<th>Values of parameters</th>
<th>Slave system</th>
<th>Synchronization</th>
<th>Theoretical results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i ) and ( \lambda_j )</td>
<td>identical (( a_i = a_j, \lambda_i = \lambda_j))</td>
<td>passive</td>
<td>intermittent</td>
</tr>
<tr>
<td>chaotic pulse-trains</td>
<td>different (( a_i \neq a_j, \lambda_i \neq \lambda_j, ; i \neq j))</td>
<td>active</td>
<td>intermittent</td>
</tr>
<tr>
<td>((N = 3))</td>
<td>periodic pulse-trains</td>
<td>passive</td>
<td>robust</td>
</tr>
<tr>
<td>pulse-trains</td>
<td>identical</td>
<td>passive</td>
<td>robust</td>
</tr>
</tbody>
</table>

Table II: The following are some future problems:

1) generalization of the theorem for different parameter values assignment and multiplexing the clock pulse train into the multiplex signal;
2) more detailed analysis of the synchronization behavior and its break;
3) analysis of the chaotic pulse train for a wider parameter range;
4) generalization and detailed analysis of the proposed stabilization technique.

Some application aspects such as the chaotic pulse-train generator and the stabilization procedure changing local stability of a system by the external periodic signal, may be developed into an information coding procedure, as discussed in [34]–[38]. It has been suggested that (chaotic) pulses play important roles in neural networks [39]–[41] and the intermittent synchronization of chaos is important in memory search dynamics [6], [42]. Our results may contribute to such chaos-based information processing and transmission schemes.

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