Analysis of Bifurcation Phenomena in a 3-Cells Hysteresis Neural Network

Kenya Jin’no, Takahiko Nakamura, and Toshimichi Saito

Abstract—This paper considers bifurcation phenomena in a simplified hysteresis neural network. The network consists of three cells and has three control parameters. We have discovered that the simple system exhibits various attractors: stable equilibria, periodic orbits, and chaos. Since the system is piecewise linear, the return map and Lyapunov exponents are calculated by using the piecewise exact solution. Using the mapping procedure, the bifurcation mechanism of stable equilibria and three kinds of bifurcation mechanisms of periodic orbits have been clarified. In addition, chaos has been analyzed by using Lyapunov exponents of the return map.

Index Terms—Bifurcation, chaos, coupled oscillator, hysteresis, piecewise linear system.

I. INTRODUCTION

Artificial neural networks (ANN’s) have been studied actively [1]–[5]. They have been used in many applications, among them, image processing, predictions, optimization, etc. ANN’s can be regarded as large nonlinear systems in which they exhibit various kinds of attractors. If the attractors and the related bifurcation phenomena can be clarified, they may contribute to develop efficient ANN’s. These systems, however, have complex nonlinearities and include many control parameters, thus, rigorous analysis is difficult.

In order to clarify the behaviors of neural networks rigorously, we have analyzed a recurrent type of ANN that includes piecewise linear hysteresis [6]–[11]. The neural network is called a hysteresis neural network and an implementation example is shown in Fig. 1(a).

The circuit equation is given by

\[
\begin{align*}
R_i C_i \frac{dv_i}{dt} &= -v_i + \sum_{j=1}^{N} R_j G_{ij} Y_j + R_l G_{il} d_l, \\
Y_i &= \begin{cases} 
+E & \text{for } v_i > -r_g h G_{il} E, \\
- E & \text{for } v_i < +r_g h G_{il} E.
\end{cases}
\end{align*}
\]

where \(v_i\) is a capacitor voltage, \(\frac{dv_i}{dt}\) is a derivative, \(R_i\) is a resistor, \(G_{ij}\) is a conductance, \(C_i\) is a capacitor, \(d_l\) is a dc term, and \(N\) is the number of cells. \(Y_i\) is an output of hysteresis whose characteristic is shown in Fig. 1(b). If \(v_i\) is switched between \(-E\) and \(+E\) if \(v_i\) hits the right threshold \(+r_g h G_{il} E\) and vice versa. Using such piecewise linear hysteresis has the following advantages.

1) Return maps and Lyapunov exponents can be calculated easily by using piecewise exact solutions.

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II. Hysteresis Network

A cell of the hysteresis neural network is shown in Fig. 1. Each cell is very simple, however, the network which consists of such cells exhibits various interesting phenomena, even if the network consists of only three cells \((N = 3)\). The attractors that are observed in laboratory measurements are shown in Fig. 2. Fig. 2(a) shows an equilibrium attractor, Fig. 2(b)–(e) shows periodic attractors, and Fig. 2(f) shows a nonperiodic attractor. Our purpose is to classify such attractors and to analyze the bifurcation phenomena. In order to analyze rigorously, we focus on the three-cell case. First, we classify the kinds of attractors and we clarify that SHN's exhibit line-expanding chaos, area-expanding chaos, and hyperchaos [14], which are confirmed by using return map attractors and Lyapunov exponents [13]. Since an SHN has piecewise linear dynamics, the map can be calculated exactly and the periodic points and related bifurcation sets are calculated theoretically, as well. These results contribute to clarification of periodic orbits and chaos in hysteresis neural networks. In addition, these results can be confirmed by laboratory experiments. We think that these results will contribute to the analysis of high-dimensional hysteresis neural networks.

In order to analyze rigorously, we focus on the following case:

\[
G_{ij} \equiv G_m \quad (i \neq j), \quad G_{ii} \equiv G_m + G_s, \quad R_i \equiv R, \quad C_i \equiv C, \quad r_b \equiv \frac{1}{g_b}, \quad G_d \equiv \frac{1}{R}, \quad d_i \equiv d, \quad N = 3. \tag{2}
\]

Using the following dimensionless variables and parameters:

\[
\tau \equiv \frac{t}{C/R}, \quad \ldots \quad \equiv \frac{d}{d\tau}, \quad y_i \equiv \frac{Y_i}{E}, \quad x_i(\tau) \equiv \frac{v_i}{E}, \quad \gamma_m \equiv RG_m, \quad \gamma_s \equiv RG_s, \quad \eta \equiv \frac{d}{E}. \tag{3}
\]

Equation (1) is transformed into

\[
\begin{align*}
x_i'(\tau) & = -x_i(\tau) + \gamma_s y_i + \gamma_m \sum_{j=1}^{3} y_j + \eta \\
y_i & = h(x_i(\tau)) = \begin{cases} +1 & \text{for } x_i(\tau) > -1 \\ -1 & \text{for } x_i(\tau) < +1 \\ \end{cases}
\end{align*} \tag{4}
\]

where \(x_i(\tau)\) is a state at \(\tau\), \(y_i\) is an output, \(\gamma_m\) is a cross connection, \(\gamma_s\) is a self feedback, and \(\eta\) is a dc term. Fig. 3 shows the characteristic of \(h(x_i(\tau))\): it is switched from \(-1\) to \(+1\) if \(x_i\) hits the right threshold \(+1\) and vice versa. Note that the term of \(\sum_{j=1}^{3} y_j\) denotes the summation value of each element of the output vector. This value is independent of \(i\). On the other hand, the term of \(\gamma_s y_i\) denotes self feedback. This value depends on \(i\). Considering the common part and the independence separately, the dynamics of this system is described by (4). Therefore, this system has three control parameters: \(\gamma_m\), \(\gamma_s\), and \(\eta\).

Since there is nonlinearity with two linear segments, we can divide the phase plane into two regions which correspond to the output of hysteresis. Note that a part of each region overlaps because a part of the domain of hysteresis output has overlap. Allowing
\( \mathbf{y} \equiv (y_1, y_2, y_3)^T \), \( \mathbf{y} \) can be classified into eight vectors

\[
\mathbf{d}_k \equiv \mathbf{y}, \quad k = \sum_{j=1}^{3} 2^{j-2} (1 + y_j)
\]

(5)

where \( \mathbf{d}_k \equiv (d_{1k}, d_{2k}, d_{3k})^T \) is a bipolar code of \( k \), e.g., \( \mathbf{d}_3 = (1, 1, -1)^T \). Using this code, the phase space can be divided into eight half spaces which correspond to the output vector \( \mathbf{d}_k \).

\[
H S_k \equiv \{ (\mathbf{x}(\tau), \mathbf{d}_k) | x_i(\tau) d_{ik} > -1 \} \quad k = 0 \cdots 7.
\]

(6)

Also, the equilibrium point \( \mathbf{p}_k \equiv (p_{1k}, p_{2k}, p_{3k})^T \), which corresponds to the half space \( H S_k \), is given from (4) by

\[
p_{ik} = \gamma_x d_{ik} + \gamma_m \sum_{j=1}^{3} d_{jk} + \eta.
\]

(7)

Equation (4) can be recast as piecewise linear equations on the half spaces which are connected by the switching of the output vector \( \mathbf{y} \). \( \dot{x}_i(\tau) = - (x_i(\tau) - p_{ik}), \quad i = 1, \cdots, 3, \quad (\mathbf{x}(\tau), \mathbf{y}) \in H S_k. \)

(8)

The trajectory moves toward an equilibrium point \( p_{ik} \). If \( p_{ik} \) exists on a hysteresis branch \( p_{ik} y_i > -1, \) \( x_i \) goes to the equilibrium point. If \( p_{ik} \) does not exist on the hysteresis branch \( p_{ik} y_i \leq -1 \), \( x_i(\tau) \) cannot reach the equilibrium point and \( y_i \) changes its sign if the \( x_i(\tau) \) reaches the threshold of hysteresis.

Definition 1: \( p_{ik} \) is said to be a real equilibrium point if \( p_{ik} y_i > -1 \) and \( p_{ik} \) is said to be a virtual equilibrium point if \( p_{ik} y_i \leq -1 \). An output \( y_i \), which corresponds to a real equilibrium point does not change its state, but an output which corresponds to a virtual equilibrium point may change its state. If \( p_{ik} \) is a real equilibrium point for all \( i \), the output vector is said to be a stable output vector. In addition, the attractor that corresponds to a stable output vector is said to be a stable equilibrium attractor. \( \square \)

If

\[
p_{ik} y_i \leq -1, \quad \text{for some} \ i \text{ and all} \ k
\]

(9)

is satisfied, a stable output vector does not exist in the system and the trajectory must oscillate.

The solution of (8) can be calculated as follows:

\[
x_i(\tau) = \exp(-\tau) x_i(0) + p_{ik} + p_{ik}
\]

(10)

where \( x_i(0) \) denotes an initial value of \( x_i(\tau) \).

Equation (10) can be recast as follows:

\[
\begin{align*}
\frac{x_1(\tau) - p_{1k}}{x_1(0) - p_{1k}} &= \frac{x_2(\tau) - p_{2k}}{x_2(0) - p_{2k}} = \frac{x_3(\tau) - p_{3k}}{x_3(0) - p_{3k}} \\
&= \exp(-\tau).
\end{align*}
\]

(11)

Namely, the trajectory is on the half line that connects the initial value and the equilibrium point. Fig. 4 shows numerical reproductions that correspond to the laboratory measurements shown in Fig. 2. Due to some errors in actual circuit parameters, coincidence is not completed between the laboratory measurement and the numerical simulation. However, the characteristics can be considered as almost coincident.

In order to analyze attractors, we define a return map. If (9) is satisfied, the trajectory goes out of the half space \( H S_k \) through a threshold and enters into another half space \( H S_j \). We define the threshold plane which is called the exit of \( H S_k \)

\[
E X_{ik} \equiv \{ (\mathbf{x}(\tau), \mathbf{y}) | x_j(\tau) = -d_{ik}, (\mathbf{x}(\tau), \mathbf{y}) \in H S_k \}. \]

(12)

We also define the set of all exits \( \mathcal{A} \equiv \{ E X_{ik} | k = 1, \cdots, 3, k = 0, \cdots, 7 \}. \)

(13)

Since the trajectory starting from \( \mathcal{A} \) must return into \( \mathcal{A} \), as long as (9) is satisfied we can define a two-dimensional (2-D) return map

\[
f: \mathcal{A} \rightarrow \mathcal{A}, \quad \mathbf{x}(\tau_j) \rightarrow \mathbf{x}(\tau_{j+1}).
\]

(14)

where \( \tau_j \) and \( \tau_{j+1} \) are, respectively, the \( j \)th moment of time when the trajectory hits the threshold plane \( \mathcal{A} \) and the \((j + 1)\)th moment of time when the trajectory returns to the plane \( \mathcal{A} \). Note that the mapping \( f \) can be calculated easily by using (10).

Next, we define

\[
q(y) = \sum_{j=1}^{3} y_j + \frac{1}{2}.
\]

(15)

Note that \( q(y) \) gives the number of times \( y_i = +1 \) occurs in the output vector \( \mathbf{y} \). Since a SHN consists of uniform cells, the attractors can be classified by the number of \( y_i = +1 \).

Definition 2: \( S_n \) denotes an equilibrium attractor such that \( q(y) \) equals \( a \).

Fig. 4(a) shows an equilibrium attractor \( S_0 \).

Definition 3: A point \( \mathbf{x}(\tau_j) \) on \( \mathcal{A} \) is said to be a periodic point with period \( m \) if

\[
f^m(\mathbf{x}(\tau_j)) = \mathbf{x}(\tau_j), \quad m \geq 1 \quad f(\mathbf{x}(\tau_j)) \neq \mathbf{x}(\tau_j), \quad m > 1
\]

(16)

where \( f^m \) denotes \( m \) times composite of \( f \).

\( f^m \) denotes a periodic orbit with period \( m \) such that \( q(y) \) alternates between \( a \) and \( b \).

Here, we consider the stability of the periodic point. If all eigenvalues of the Jacobi matrix of \( f^m(\mathbf{x}_n) \) are within the unit circle, the periodic point is stable. The calculation algorithm of the Jacobi matrix is explained in Section IV. If there is a stable periodic point with period \( m \), the attractor becomes a periodic orbit. Some examples of periodic attractors are shown in Fig. 4(b)–(e). In this figure, the break points of the trajectory correspond to the periodic points, for example, Fig. 4(b) has six break points, hence, it is period six. The time-domain waveform of Fig. 4(b) is shown in Fig. 5. In this case, \( \mathbf{x} \) and \( \mathbf{y} \) return to the same states after six times switching and \( q(y) \) takes the value of one or two.

Definition 4: The attractor is said to be a nonperiodic orbit if it is neither a periodic orbit nor a stable equilibrium point. \( N P_n \) denotes a nonperiodic orbit such that \( q(y) \) alternates between \( a \) and \( b \).

An example of a nonperiodic attractor is shown in Fig. 4(f). In Section V we consider such nonperiodic attractors by using Lyapunov exponents.
equilibria. If an SHN has a stable equilibrium attractor

A. Bifurcation of Stable Equilibria

In this section we consider the bifurcation phenomena of stable equilibria. If an SHN has a stable equilibrium attractor \( S_{i}(\omega) \), the equilibrium point \( p_{i} \) is given as follows:

\[
p_{i} = \gamma_{m} (2q(y) - N) + \gamma_{s} y_{i} + \eta.
\]  

Based on Definitions 1 and 2, the existence condition of \( S_{i}(\omega) \) is derived

\[
(\gamma_{m} (2q(y) - 3) + \gamma_{s} y_{i} + \eta) y_{i} > -1, \quad \text{for all } i.
\]  

The bifurcation set of \( S_{i}(\omega) \) is the boundary of the above condition. Fig. 6 shows an example of a calculated result of a two-parameter bifurcation diagram of stable equilibria.

B. Bifurcation of Periodic Orbits

Next, we consider bifurcation phenomena of periodic orbits. Fig. 7 shows a one-parameter bifurcation diagram by plotting \( x_{2} \) when \( x_{1} \) hits the right threshold of hysteresis \( x_{1} = 1, y_{1} = -1 \). In this figure, we can see the following transition of attractors: \( P_{12}^{6} \rightarrow P_{12}^{8} \rightarrow N P_{12} \rightarrow P_{12}^{12} \rightarrow P_{12}^{6} \). From Fig. 7 there are various bifurcation mechanisms. We will classify the bifurcation mechanisms into three cases in relation to periodic orbits.

1) Changing Stability of the Periodic Point: The bifurcation of the periodic orbit by changing from a stable periodic point to an unstable periodic point is well known as a general bifurcation phenomena, e.g., period-doubling bifurcation and Hopf bifurcation. Similar bifurcation can be observed in SHN’s. In Fig. 7, the bifurcations of \( P_{12}^{6} \rightarrow P_{12}^{8} \) and \( P_{12}^{12} \rightarrow N P_{12} \) are caused by this mechanism. The periodic point of \( P_{12}^{12} \) is a stable periodic point while the attractor exhibits \( P_{12}^{12} \). The periodic point of \( P_{12}^{12} \), however, becomes an unstable periodic point while the attractor exhibits \( P_{12}^{12} \).

2) Simultaneous Switching: A characteristic bifurcation phenomenon of the hysteresis neural network is caused by a simultaneous switching. The location of the equilibrium points of this system depends on the output vector, and then the switching of the output vector plays a very important role in the bifurcation phenomena. Here, we explain this bifurcation mechanism by using Fig. 8, which shows a phase space. Each circle in Fig. 8(a) and (b) corresponds to the periodic points of a periodic attractor. We consider the case where the periodic attractor of Fig. 8(a) bifurcates to the attractor of Fig. 8(b). Since two or more periodic points do not exist on the hysteresis threshold in Fig. 8(a) and (b), the switching does not occur at the same time. The periodic point of Fig. 8(a) indicates that \( x_{1} \) hits the threshold before \( x_{1} \). The situation of Fig. 8(b) indicates that \( x_{1} \) hits the threshold faster than \( x_{1} \). The bifurcation is caused by the simultaneous switching, as shown in Fig. 8(c). In this case, \( x_{1} \), and \( x_{1} \) exist on the hysteresis threshold and hit the threshold simultaneously. Therefore, we refer to the bifurcation as simultaneous switching. In Fig. 7, \( P_{12}^{12} \rightarrow P_{12}^{6} \) and \( P_{12}^{12} \rightarrow N P_{12} \) are caused by this bifurcation mechanism.

3) Quasi-Simultaneous Switching: The hysteresis neural network has another bifurcation mechanism that is similar to simultaneous switching. We explain the bifurcation mechanism by using Fig. 9. We consider the case where the periodic attractor of Fig. 9(a) bifurcates to the attractor of Fig. 9(b). These situations are similar to the case of Fig. 8. Fig. 9(c) shows the state of the bifurcation point. Since \( x_{1} \) is located on the right threshold when the output of \( x_{1} \) changes its sign, \( x_{1} \), and \( x_{1} \), have the same value. The process until bifurcation is like simultaneous switching. Therefore, we call this bifurcation quasi-simultaneous switching. In Fig. 7, \( P_{12}^{12} \rightarrow P_{12}^{6} \) is caused by this bifurcation mechanism.

Fig. 10 shows an example of a two-parameter bifurcation diagram for \( \gamma_{m} = 5 \).

IV. Calculation Algorithm for Bifurcation Sets of Periodic Orbits

First, we explain the case where bifurcation sets can be obtained by the stability of the periodic point. The stability of the periodic point is calculated by using the eigenvalues of the Jacobi matrix along the periodic orbit. The Jacobi matrix is given by

\[
Df^{m} = \frac{\partial f^{m}(x_{m}(\tau_{n}))}{\partial x_{m}(\tau_{n})} = \frac{\partial x_{m}(\tau_{n})}{\partial x_{m}(\tau_{n-1})} \frac{\partial x_{m}(\tau_{n-1})}{\partial x_{m}(\tau_{n-2})} ... \frac{\partial x_{m}(\tau_{1})}{\partial x_{m}(\tau_{0})}
\]  

where \( Df^{m} \) is a \( 2 \times 2 \) Jacobi matrix along the periodic orbit. The periodic points are calculated by using (16). Substituting these periodic points into (19) we can obtain \( Df^{m} \). The eigenvalues of the
The Jacobi matrix are obtained by

\[ \lambda I - D f^m = 0. \]  

(20)

Let \( \lambda_1 \) and \( \lambda_2 \) denote the eigenvalues such that \( |\lambda_1| \geq |\lambda_2| \). If \( 0 < |\lambda_1| < 1 \) implies a stable periodic point, \( |\lambda_1| > 1 \) implies an unstable periodic point, and \( |\lambda_1| = 1 \) implies a bifurcation point between stable and unstable periodic points. The bifurcation set can be calculated by using regula falsi for a control parameter.

We show a calculation example of the periodic point for \( P_{12}^{m} \) whose time-domain waveform is shown in Fig. 5. Equations for solving \( f^m(x(\tau_0)) = x(\tau_0) \) are obtained by substituting the series of \( x \) and \( p_x \) into (11)

\[
\begin{align*}
\frac{x_1(\tau_1) - (\gamma_m - \gamma_s + \eta)}{x_1(\tau_0) - (\gamma_m - \gamma_s + \eta)} &= -1 - (\gamma_m + \gamma_s + \eta) \\
\frac{x_2(\tau_0) - (\gamma_m + \gamma_s + \eta)}{x_2(\tau_0) - (\gamma_m + \gamma_s + \eta)} &= \frac{x_3(\tau_1) - (\gamma_m + \gamma_s + \eta)}{1 - (\gamma_m + \gamma_s + \eta)} \\
\frac{1 - (-\gamma_m - \gamma_s + \eta)}{x_1(\tau_1) + (-\gamma_m - \gamma_s + \eta)} &= \frac{x_2(\tau_2) - (-\gamma_m - \gamma_s + \eta)}{1 - (-\gamma_m - \gamma_s + \eta)} \\
\frac{x_3(\tau_2) - (-\gamma_m - \gamma_s + \eta)}{x_3(\tau_1) - (-\gamma_m - \gamma_s + \eta)}
\end{align*}
\]

(21)

Since all cells have equal parameter values, \( x_2(\tau_2) \) and \( x_3(\tau_2) \) take the same value as \( x_1(\tau_0) \) and \( x_2(\tau_0) \), respectively. Then we can obtain the periodic points by using (21). For example, if we give the parameters for Fig. 4 \( \gamma_m = -5 \), \( \gamma_s = -2 \), and \( \eta = 0 \), we can calculate the periodic points \( x_1(\tau_0) \approx 0.216 \) and \( x_2(\tau_0) \approx 0.348 \). Substituting them into (20), the eigenvalue is given by \( |\lambda_1| \approx 0.59 \), that is, the periodic point \( P_{12}^{m} \) is stable. If we give the parameters for Fig. 7 \( \gamma_m = -5 \) and \( \gamma_s = -2 \) we can confirm that \( |\lambda_1| = 1 \) is satisfied for \( \eta \approx -3.555 \). The value of this parameter gives the bifurcation point \( P_{12}^{m} \rightleftharpoons P_{12}^{r} \).

Next, we explain the calculation algorithm for simultaneous switching. We consider a sequence of periodic points, including the simultaneous switching state. The sequence can exist only if the following equation is satisfied:

\[ f'(x(\tau_0)) - x(\tau_0) = 0 \]

(22)

where \( s \) denotes the period of periodic points.

Since the function \( f'(x(\tau_0)) \) includes \( 2s + 2 \) parameters, if two of them are fixed the other parameters can be calculated.

Here we show a calculation example of the bifurcation set for simultaneous switching. The bifurcation is \( P_{12}^{m} \rightleftharpoons P_{01}^{m} \) and the series
of \( x \) and \( y \) are as follows:

\[
\begin{array}{cccccc}
    & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \\
y_1 & 1 & -1 & -1 & 1 & \\
y_2 & -1 & 1 & -1 & -1 & \\
y_3 & -1 & -1 & 1 & -1 & \\
x_1 & 1 & -1 & x_1(\tau_2) & 1 & \\
x_2 & x_2(\tau_3) & 1 & -1 & x_2(\tau_0) & \\
x_3 & -1 & 1 & x_3(\tau_1) & 1 & -1 & \\
q(y) & 1 & 1 & 1 & 1 & \\
\end{array}
\]

Note that \( q(y) \) becomes the constant \( q(y) = 1 \). Therefore, (22) is obtained as follows:

\[
-1 = \frac{\gamma_m + \gamma_x + \eta}{\gamma_m + \gamma_x + \eta} = \frac{1}{\gamma_m(\tau_0) - \gamma_m(\gamma_x + \eta)} = \frac{\gamma_m(\tau_1) - \gamma_m(\gamma_x + \eta)}{-1 - \gamma_m(\gamma_x + \eta)}
\]

(23)

In this case, \( x_3(\tau_1) = x_3(\tau_0) \) because all cells have the same parameters. Substituting \( x_3(\tau_1) = x_2(\tau_0) \) for (23), \( \eta \) is calculated by

\[
\eta = \frac{3\gamma_m - \gamma_x}{3} - \frac{\sqrt{4\gamma_x^2 - 3}}{3}.
\]

(24)

Substituting \( \gamma_m = -5 \) and \( \gamma_x = -2 \), we can obtain the value of the bifurcation point precisely: \( \eta \approx -5.535 \).

V. NONPERIODIC ORBITS

In order to classify the nonperiodic attractors in more detail, we use Lyapunov exponents [13] of the 2-D return map \( f \) rather than eigenvalues. Letting \( \mu_1 \) denote the maximum \( i \)-th dimensional Lyapunov exponent given by

\[
\mu_1 = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \ln \| D_f e_i \|
\]

(25)

where \( D_f \) is the Jacobi matrix of the 2-D return map \( f \) at \( \tau_j \) and \( e_i \) is a normalized base. We have confirmed that Lyapunov exponents have converged reasonably in \( 10^4 \) iterations, hence, we set \( n = 5 \times 10^4 \) in (25) in actual calculations [10].

If a trajectory started from \( x(\tau_{j-1}) \) goes to \( x(\tau_j) \), the Jacobi matrix of \( D_f \) is given by

\[
D_f = \frac{\partial x(\tau_j)}{\partial x(\tau_{j-1})}.
\]

(26)

The basic attractors can be characterized by the maximum one-dimensional (1-D) Lyapunov exponent \( \mu_1 \). If the system exhibits periodicity, \( \mu_1 \) has a negative value. If the system exhibits chaos, \( \mu_1 \) has a positive value. Moreover, the chaotic attractors can be classified as follows. If \( \mu_2 \) has a negative value the system exhibits line-expanding chaos. If \( \mu_2 \) has a positive value the system exhibits area-expanding chaos. Especially, if \( \mu_2 > \mu_1 > 0 \) is satisfied, the attractor is said to be hyperchaos [14]. Fig. 11 shows the diagram of Lyapunov exponents for \( \gamma_m = -5.0 \) and \( \gamma_x = -2.0 \). We have confirmed that line-expanding chaos, area-expanding chaos, and hyperchaos have been observed. In Fig. 12, we can see some features from their return map attractors. Fig. 12(a) shows a line-expanding chaos attractor, Fig. 12(b) shows an area-expanding chaos attractor, and Fig. 12(c) shows a hyperchaos attractor. We have confirmed that the hyperchaos attractor fills up some regions more densely than others. Fig. 12(d) is the attractor near the bifurcation point between line-expanding chaos and the periodicity \( P_{02} \). We have named \( P_{02} \) bifurcates to \( \gamma_x \) as changing stability of the periodic point and the attractor shows the characteristic of Hopf-bifurcation.

VI. CONCLUSIONS

We have analyzed bifurcation phenomena from a SHN consisting of three cells. The SHN exhibits various attractors: stable equilibria, periodic orbits and nonperiodic orbits. We have clarified the bifurcation phenomena of stable equilibria and three kinds of bifurcation in periodic orbits: changing stability of the periodic point, simultaneous switching, and quasi-simultaneous switching. Nonperiodic orbits have been classified by using Lyapunov exponents of the return map. Then we have confirmed that the SHN exhibits line-expanding chaos, area-expanding chaos and hyperchaos. In addition, these results have been calculated by using an exact solution and have been confirmed by laboratory measurements. Now we are trying to develop these results for higher dimensional systems.

REFERENCES


A Low-Power and High-Speed Dynamic PLA
Circuit Configuration for Single-Clock CMOS

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Abstract—Certain logic functions such as the control units of VLSI processors are difficult to implement by random logic. Since the programmable logic arrays (PLA’s) can implement almost any Boolean function, they have become popular devices in the realization of both combinational and sequential circuits. We present a low-power high-speed complementary-metal-oxide semiconductor (CMOS) circuit implementation of NOR-NOR PLA using a single-phased clock. Buffering static NAND gates are inserted between the NOR planes to erase the racing problem and shorten the duration of glitches such that the dynamic power is reduced in addition to the low static power dissipation, no ground switch, no charge sharing, and zero offset.

Index Terms—High-speed, low-power, NOR-NOR PLA, single clock.

I. INTRODUCTION

PLA’s can be implemented by either static or dynamic styles. The choice is dependent on the timing and power strategies. Modern CAD tools are required to support the integration of commonly used single-phased edge-triggered basic elements [1], including programmable logic arrays (PLA’s). Before the discussion of the proposed PLA design, the shortcoming of several PLA design methods are listed as follows.

Pseudo-NMOS [5]: It is the simplest design style to realize PLA’s. The main disadvantage of this approach is the dc-path dissipation. In addition, because of the ratioed design the PMOS and NMOS have to be enlarged dually when the pull-up time is critical. Meanwhile, the ratioed design will reduce the speed.

Dynamic NOR-NOR [5], [4]: The major problem of this type of logic is the racing problem when two dynamic logic gates are cascaded in series. There is a possibility that the output of the first gate wrongly turns the second gate ON or OFF such that the final result is incorrect. Thus, it is necessary to generate a delayed clock for the second gate in order to prevent the racing problem. This will reduce the operation speed. In addition, the ground switch will produce a large parasitic capacitance which certainly reduces the speed.

Domino [5]: In domino-logic design, the gates are all precharged, and connected to the next stage through inverters. Although the SOP domino circuits are excellent with regard to power saving, the serial NMOS’s of the front AND plane will cause a large pull-down delay. In addition, the serial NMOS’s could cause charge-sharing problems.

Dhong’s Design [3]: Dhong et al. proposed a PLA design approach which employs a precharged OR array and a charge-sharing AND array to eliminate the ground switch of the second gate. Since the charge sharing is used, the output voltage \( V_{OD} \) can only reach approximately \( 3.0 \) V when \( V_{OD} \) is \( 5.0 \) V. It cannot provide the full swing of the voltage aside from the low-noise margin problem. As well, a delayed clock is needed in order to prevent the racing problem.

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