Synchronization of Chaos and Its Itinerancy from a Network by Occasional Linear Connection

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Abstract—This paper proposes a network of continuous-time chaotic cells and considers its dynamics. The cell includes a bipolar hysteresis whose thresholds vary periodically. The cell exhibits chaos and various stable periodic orbits. We have classified these phenomena in a bifurcation diagram and have clarified basic generation mechanism of these phenomena. The network is constructed by using the Occasional Linear Connection method that connects the cells occasionally by using a sampled state of each cell. The network exhibits various phenomena: synchronization of stable periodic orbits, synchronization of chaos, and chaotic itinerancy. We have classified these phenomena and have clarified their existence condition.

These results are guaranteed theoretically and are verified in the laboratory.

I. INTRODUCTION

Recently, networks of chaotic elements have been studied with great interests. Many of them are fundamental models of physical and biological systems [1]–[12]. These networks exhibit various interesting phenomena, among them synchronization of chaos and chaotic itinerancy. Roughly speaking, synchronization of chaos implies temporally chaotic and spatially ordered phenomenon [13], [14], and chaotic itinerancy implies an aperiodic switching among plural temporary phenomena [2], [3], for example, chaotic switching between in-phase and antiphase synchronization. (In Section IV, we give some comments on the chaotic itinerancy.) Such phenomena have been confirmed in various systems, e.g., chaotic neural networks [4] and globally coupled maps (GCM’s) [1]–[3]. Chaotic neural networks may contribute to realize effective memory searches [5] and to solve complex optimization problems [6]. Referring to behavior of the GCM, the chaotic itinerancy may play an important role for information processing functions, e.g., clustering, coding, and learning [1]. However, there are many open problems for analysis and synthesis of the networks of chaotic elements: classification of the phenomena, recognition of the phenomena, and implementation of the network.

In order to approach these problems, this paper proposes a network of simple continuous-time chaotic cells and considers its dynamics. In Section II, we propose the cell that includes a bipolar hysteresis whose thresholds vary periodically. The dynamics of the cell can be simplified into a one-dimensional return map that is described explicitly. In Section III, using the return map, we clarify that the cell exhibits various kinds of periodic orbits and those of chaotic attractors. We have classified these phenomena in a bifurcation diagram. In Section IV, we propose a network of the cells by using a novel version of the Occasional Linear Connection (OLC) [10]: an occasional connection method using a sampled state of each cell. The network exhibits various kinds of synchronization of chaos, various kinds of stable periodic orbits, and chaotic itinerancy. We have classified these phenomena and have clarified their existence condition based on the bifurcation diagram from the cell. These phenomena are guaranteed theoretically and verified in the laboratory. Section V concludes the paper.

In [10], we have proposed another version of the OLC that can realize in-phase, lagging-phase, and leading-phase synchronization of chaos. However, it discusses neither chaotic itinerancy nor bifurcation phenomena. In [15], we have considered another chaotic cell using an implicit return map, however, we have discussed neither rigorous proof of chaos generation nor classification of chaos.

II. CELL

We propose a cell, a unit subsystem of the OLC network:

\[ \lambda \dot{x} = px + y, \quad y = h(x, \tau) \]  

where \( x \) is a state, \( y \) is an output, \( \tau \) is a normalized time, \( \dot{x} \equiv dx/d\tau \), \( p \geq 1 \), and \( n \) is a positive integer. Also, \( h(x, \tau) \) is a time-variant hysteresis whose thresholds vary periodically as shown in Fig. 1(a): \( h(x, \tau) \) is switched from 1 to \(-1\) (respectively, \(-1\) to 1) if \( x \) hits the right threshold \( T^+ h(\tau) \) (respectively, the left threshold \(-T^- h(\tau)\)), where \( x \) is continuous at the switching moment. \( x \) corresponds to a capacitor voltage in a circuit model shown afterward. Two parameters \( p \) and \( \lambda \) control equilibrium point and time constant, respectively. This system has odd symmetry with respect to \( x = 0 \).

In order to understand basic dynamics of the cell, we consider the case of \( a = 1 \). In this case, \( h(x, \tau) \) is time-invariant and the cell behaves as below (see Fig. 1(b)).

1. For \( p \leq -1 \), \( x \) converges to one of two equilibrium points, \( \frac{1}{p} \) and \(-\frac{1}{p} \): the cell is a bistable multivibrator.
2. For \(-1 < p < 1 \), the cell oscillates with period \( 2T_p = \frac{2\pi}{\ln|2p|} \): the cell is an astable multivibrator.
3. For \( p > 1 \), \( x \) diverges to either \( \infty \) or \(-\infty \).

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Fig. 1. Basic dynamics of the chaotic cell. (a) Time-variant hysteresis. (b) Dynamics on the phase plane.

Fig. 2. Typical orbits. The upper and the lower dotted lines denote $J_1$ and $J_2$, respectively.

Next, we consider the case where $0 < p < 1$ and $1 \ll a$. The orbit started from $(x, y, \tau) = (1, 1, 1)$ increases as curve (a) in Fig. 2. At $\tau = 2$, the right threshold changes from $a$ to $1$ and $y$ changes from $1$ to $-1$. Letting $x(2) = x_m$, we have $x_m = (1 + \frac{1}{p})c - \frac{1}{p}$. If $x_m < \frac{1}{p}$, the state $x$ decreases and reaches $-1$ at $\tau = 2 + T_q$, where $T_q = \frac{2}{p} \ln \frac{1 + p}{1 - p}$. The quantities $T_p$, $x_m$, and $T_q$ play important roles in the following analysis. Note that the output $y$ does not change its sign for $2n - 1 \leq \tau < 2n$ and the parameter $a$ does not affect the system dynamics if

$$-1 < p < 1, \quad x_m < \frac{1}{p} < a. \quad (2)$$

Hereafter, we assume (2) and omit $\alpha$; if either $|x| = 1$ for $2n - 2 \leq \tau \leq 2n - 1$ or if $|x| > 1$ at $\tau = 2(n - 1)$ is satisfied, $y$ can change its sign.

In order to consider the dynamics of the cell, let $J_1 = \{(x, y, \tau) \mid x = 1, y = 1, 0 \leq \tau\}$ and let $J_2 = \{(x, y, \tau) \mid x = -1, y = -1, 0 \leq \tau\}$. As shown in Fig. 2, the orbit started from $J_1$ at $\tau = \tau_0$ must hit $J_2$ and $J_1$ alternately. Letting $\tau_n$ denote the $n$th hit time and noting that $J_1$ is symmetric to $J_2$ with respect to $x = 0$, the hit time sequence $\{\tau_n\}$ is given by

$$\tau_{n+1} = f(\tau_n) = \begin{cases} 
\tau_n + T_p, & \text{for } 2(n - 1) \leq \tau_n \leq 2n - 1 \\
\frac{2}{p} \ln \left(\frac{p + 1}{p - 1}\right) + 2n, & \text{for } 2n - 1 < \tau_n < 2n.
\end{cases} \quad (3)$$

An example of $f$ is shown in Fig. 3. For simplicity, we assume

$$f(1) = T_p + 1 < 3, \quad f(1+) = T_q + 2 < 5. \quad (4)$$

In this case we can define the return map $F$ as shown in Fig. 3:

$$z_{n+1} = F(z_n), \quad z_n \in I \equiv [0, 3]$$

$$z_{n+1} = \begin{cases} f(z_n), & \text{for } 0 \leq z_n \leq 1 \\
f(z_n) - 2, & \text{for } 1 < z_n \leq 3.
\end{cases} \quad (5)$$
The return map has one discontinuous point and one extremum

$$F(1) = T_p + 1, \quad F(1_+) = T_q, \quad F(2_-) = F(2) = T_p.$$  

If we construct a modulo 2 map of $f$ as is discussed in [10], the map becomes a complicated form that has three discontinuous points: the return map $F$ is convenient for the system analysis. Note that the sequence $\{z_n\}$ of $F$ corresponds uniquely to hit time sequence $\{\tau_n\}$ because (3) can be recast as

$$\tau_{n+1} = \begin{cases} F(\tau_n), & \text{for } 0 \leq \tau_n < 1 \\ F(\tau_n - (2m - 2) + 2m), & \text{for } 2m - 1 \leq \tau_n < 2m + 1 \end{cases}$$  

where $m$ is a positive integer. Hereafter, we focus on the following parameter range:

$$(\lambda, p) \in \{ (\lambda, p) | -1 < p < 1, \quad 0 < T_p < 2, \quad 0 < T_q < 3, \quad x_m < \frac{1}{|f|} \}.$$  

As the parameters vary, the return map exhibits various phenomena, as shown in Fig. 4. These phenomena are analyzed in Section III.

Fig. 5 shows an implementation example of the cell, where the terminal $c$ is grounded. (In Section IV, the terminal $c$ is used to construct a network.) In this figure, $-G$ is a linear negative conductor whose implementation example is shown in [10]. The periodic square signal, $S(t) = V_{\text{sgn}}(\sin(\frac{\pi}{T}t))$, is applied to the switch SW. The signal $S(t)$ controls the hysteresis thresholds and realizes time-variant hysteresis

$$H(v, t) = \begin{cases} E, & \text{for } v \leq TH(t) \\ -E, & \text{for } v \geq -TH(t) \end{cases}$$  

$$TH(t) = \begin{cases} V_L, & \text{for } S(t) = V \quad (\text{SW is on}) \\ E, & \text{for } S(t) = -V \quad (\text{SW is off}) \end{cases}$$  

where $V_L \equiv \frac{V}{r + R}$ and $E$ is a saturation voltage of the Op.Amp. $H$ is switched from $E$ to $-E$ (respectively, $-E$ to $E$) if $v$ hits the right threshold $TH(t)$ (respectively, the left
threshold \(-\text{TH}(t)\)). The circuit equation is given by
\[
\text{RC} \frac{dv}{dt} = (\text{RG} - 2)v + H(v, t).
\] (8)

Using the following transformations, we obtain (1).
\[
\begin{align*}
\tau &= \frac{t}{T}, \\
x &= \frac{v}{V_L}, \\
h(x, \tau) &= \frac{1}{E} H(V_L x, T \tau), \\
a &= \frac{r_a + r}{r_a}, \\
\lambda &= \frac{\text{RC}}{a T}, \\
p &= \frac{1}{a} (\text{RG} - 2).
\end{align*}
\] (9)

Fig. 6 shows laboratory data from an implementation circuit: the chaotic orbit \(v\) in Fig. 6(a) (respectively, Fig. 6(b)) corresponds to the chaotic attractor in Fig. 4(a) (respectively, Fig. 4(c)); and stable orbits \(v\) and \(v'\) in Fig. 6(c) corresponds to periodic attractors in Fig. 4(e). Note that (1) has odd symmetry with respect to \(x = 0\); one sequence from the return map corresponds to two symmetric orbits from the cell.

### III. Chaos and Bifurcation

In this section, we consider chaos and bifurcation from the cell. First, we define an attractor.

**Definition 1** An infinite sequence \(\{z_0, z_1, \ldots\}\) from the return map \(F'\) is said to be an orbit of \(z_0\). A subset \(I_a\) in \(I\) is said to be an attractor if \(F(I_a) = I_a\) and there exists some subset \(I_c \supset I_a\) such that any orbit started from \(I_c\) eventually enters into the closure of \(I_a\).

The attractor \(I_a\) characterizes stationary phenomena. Referring to [16], we have:

**Theorem 1** \(F\) exhibits chaos if \((\lambda, p) \in C_0 \equiv \{ (\lambda, p) \mid 0 < p < 1, 0 < T_p < 2, 0 < T_q < 3 \}\).

**Proof:** Differentiating (3), we obtain
\[
DF(z) \equiv \frac{dF(z)}{dz} = \begin{cases} 
eg\frac{\epsilon^z}{(\lambda + 1)} & \text{for } 1 < z < 2 \\ 1 & \text{otherwise.} \end{cases}
\] (10)

If \(p > 0\), (10) guarantees \(|DF| > 1\) for \(1 < z < 2\). If \(0 < T_p < 2\), there exists some finite positive integer \(m(z_0)\) such that \(1 < F^m(z_0) < 2\) for any \(z_0\) in \(I\), where \(F^m\) denotes the \(m\)-fold composition of \(F\). Letting \(n\) be the maximum value of \(m(z_0)\) for \(z_0\), \(|DF^n| > 1\) is satisfied for \(0 \leq z_0 \leq 3\). In this case, [16] says that \(F\) is ergodic and has a positive Lyapunov exponent: \(F\) exhibits chaos.

If \(F\) exhibits chaos, almost all orbits behave chaotically in some attractor \(I_a\); hence we refer to such attractor \(I_a\) as a chaotic attractor. Then we show some basic results for the chaotic attractor.

**Theorem 2** We divide the parameter set \(C_0\) into the following three subsets, as shown in Fig. 7:
\[
\begin{align*}
C_1 &= \{ (\lambda, p) \mid 0 < p < 1, 1 < T_p < 2, T_q < 3 \} \\
C_2 &= \{ (\lambda, p) \mid 0 < p < 1, T_p \leq 1, 2 < T_q < 3 \} \\
C_3 &= \{ (\lambda, p) \mid 0 < p < 1, T_q \leq 2 \} \\
C_1 \cup C_2 \cup C_3 &= C_0.
\end{align*}
\]
If \( (\lambda, p) \in C_1 \) then \( I_a \subseteq I_1 \equiv [T_p, F(T_p)] \) and \( 2 \in I_a \) are satisfied, that is, the chaotic attractor includes \( z = 2 \).

(2) If \( (\lambda, p) \in C_2 \) then \( I_a \subseteq I_2 \equiv [T_q, T_q] \) is satisfied, that is, the chaotic attractor includes both \( z = 1 \) and \( 2 \).

(3) If \( (\lambda, p) \in C_3 \) then \( I_a \subseteq I_3 \equiv [F(T_q), T_q] \) and \( 1 \in I_a \) are satisfied, that is, the chaotic attractor includes discontinuous point \( z = 1 \).

Proof: Fig. 4(a)–(c) corresponds to the above three cases, respectively. Referring to [16], this theorem can be proven immediately from these figures. Hence, we confirm the proof only for \( (\lambda, p) \in C_3 \). 0 < \( p \) guarantees \( DF < -1 \) for \( 1 < z < 2 \) and \( T_q < 2 \) guarantees that \( F \) is monotone for \( 1 < z < T_q \). For \( z_0 \in [0, 1] \cup [T_q, 3] \), the orbit started from \( z_0 \) enters into \( I_3 \), and after that it can not escape from \( I_3 \). In this case [16] guarantees that there exists a chaotic attractor \( I_a \) that includes the discontinuous point \( z = 1 \).

As we discuss in Section IV, the discontinuous point plays an important role for generation of chaotic itinerancy. For \( (\lambda, p) \in C_2 \), it is hard to clarify whether or not \( I_a \) includes the discontinuous point. As far as numerical results concern, \( 1 \in I_a \) is dominant in \( C_2 \), however, we have observed the case where \( I_a \) does not include the discontinuous point as shown in Fig. 4(c). In the latter case, \( F \) may exhibit Islands [17] and we are trying to analyze such phenomena.

Next, we define the periodic attractor.

**Definition 2** A point \( z_a \) is said to be an \( m \)-periodic point if \( F^m(z_a) = z_a \) and \( F^k(z_a) \neq z_a \) for \( 1 < k < m \). A point \( z_a \) is said to be a 1-periodic point (or fixed point) if \( F(z_a) = z_a \). A sequence of \( m \)-periodic points \( \{F(z_a), \ldots, F^m(z_a)\} \) is said to be an \( m \)-periodic sequence. An \( m \)-periodic sequence is said to be stable (respectively, unstable) if \( |DF^m(z_a)| \leq 1 \) (respectively, \( |DF^m(z_a)| > 1 \)). We refer to a stable \( m \)-periodic sequence as an \( m \)-periodic attractor. The cell eventually exhibits a stable \( m \)-periodic orbit if the return map \( F \) exhibits an \( m \)-periodic attractor.

Then we have:

**Theorem 3** (Proof is in the Appendix.)

As shown in Fig. 7, we define some subsets

\[
\begin{align*}
P_m &\equiv \{ (\lambda, p) : -1 < p < 0, 0 < T_p < 2 \} , \\
P'_m &\equiv \{ (\lambda, p) : -1 < p < 0, T_p < 1, T_q > 1 \} , \\
&\text{for } m = 1 ; \\
P_m' &\equiv \{ (\lambda, p) : -1 < p < 0, \\
&\quad T_q + (m - 2)T_p < 1 \leq T_q + (m - 1)T_p , \\
&\quad 1 < F(1 + T_p) + (m - 1)T_p , \\
&\quad \exists \quad \text{for } m \geq 2 .
\end{align*}
\]

1) If \( (\lambda, p) \in P_m \) then \( F \) exhibits an \( m \)-periodic attractor.

2) If \( (\lambda, p) \in P'_m \) then \( F \) exhibits either an \( m \)-periodic or an \( (m + 1) \)-periodic attractor. We can observe one of them depending on the initial state.

Fig. 4(e)–(g) shows examples of periodic attractors and Fig. 6 shows laboratory measurements. Note that this theorem is useful also for understanding bifurcation phenomenon from the OLC network.

IV. NETWORK BY OCCASIONAL LINEAR CONNECTION

In this section, we propose a network of the cells by using OLC:

\[
\begin{align*}
\chi_i &\equiv \begin{cases} \mu x_i + y_i, & \text{for } 2(n-1) \leq \tau \leq 2n-1 \\
px_i + y_i + \sum_{j=1}^{N} W_{ij} x_j (2n-1), & \text{for } 2n-1 < \tau < 2n 
\end{cases} \\
&= \sum_{j=1}^{N} W_{ij} x_j (2n-1) + y_i \\
&= K x_i + y_i .
\end{align*}
\]  

(11)

where \( x_i \) is the \( i \)-th state, \( y_i \equiv h(x_i; \tau) \) is the \( i \)-th output, \( i \in \{1,2, \ldots , N\} \), \( N \) is the number of cells, and \( n \) is a positive integer. Note that such a subscript \( i \) is often used as a discrete spatial coordinate in neural networks [4]–[8]. Also, the connection \( W_{ij} \) is defined by

\[
W_{ij} = \frac{K}{N} - K, \quad W_{ij} = \frac{K}{N}, \quad K = \frac{p}{1 - e^{-\frac{p}{8}}} ,
\]

for \( i \neq j \).  

(12)

During the first half period, \( 2(n-1) \leq \tau \leq 2n-1 \), all cells are free-running, and during the second half period, \( 2n-1 < \tau \leq 2n \), they are connected to each other by using the sampled state at \( \tau = 2n-1 \) (see Fig. 8). Hereafter, we refer to this network as the OLC network (OLCN). Note that \( W_{ij} \) depends on outputs \( y_i \) and \( y_j \). For convenience, we recast (11) as the following:

\[
\lambda \chi_i = \begin{cases} \mu x_i + y_i, & \text{for } 2(n-1) \leq \tau \leq 2n-1 \\
px_i + y_i + \Delta y_i, & \text{for } 2n-1 < \tau < 2n
\end{cases}
\]

(13)

\[
\Delta y_i = K(y_i \bar{X}(2n-1) - x_i(2n-1)), \quad \bar{X} = \frac{1}{N} \sum_{j=1}^{N} y_j x_j,
\]
During the second half period, the output $y_i$ is constant and the solution is given as the following:

$$x_i(\tau) + \frac{y_i}{p} + \Delta y_i = \exp\left(\frac{p}{2}(\tau - 2n + 1)\right)\left(y_i x_i(2n - 1) + \frac{y_i}{p} + \Delta y_i\right),$$

for $2n - 1 < \tau < 2n$.

Then the state at the period end, $\tau = 2n$, is given by

$$x_i(2n) + \frac{y_i}{p} = \exp\left(\frac{p}{2}x_i(2n - 1) + \frac{y_i}{p}\right).$$

That is, $x_i(2n) = y_i(2n - 1)\hat{X}(2n)$ is satisfied: all states have the same magnitude at $\tau = 2n$.

Here, we consider an ideal case: noise does not exist. If $y_i(2n - 1) = 1$ for all $i$, the OLCN exhibits either synchronization of chaos or that of stable periodic orbits. Note that “$y_i(2n - 1) = 1$ for all $i$” and “$y_i(2n - 1) = -1$ for all $i$” give different kinds of in-phase synchronous states, because (1) has odd symmetry with respect to $x = 0$. Noting that a vector $(y_1, \ldots, y_N)$ can have $2^N$ kinds of values, the OLCN exhibits $2^N$ kinds of synchronization.

Next, we consider a realistic case: small noise exists. In this case, repeating the OLC may hold the synchronous phenomena. In order to confirm the above suggestion, we propose an implementation example as shown in Fig. 9: it consists of $N$ pieces of the cells, sample & holds circuits (S/H) (see Fig. 9(b)), a mean-state generator [10] (see Fig. 9(c)), and OLC signal generator (see Fig. 9(d)). In this figure, the mean-state generator outputs

$$Y_i((2n - 1)T) = \frac{1}{N} \sum_{j=1}^{N} Y_j((2n - 1)T)$$

where $Y_j$ denotes the $j$th state, $Y_i$ denotes the $i$th output and $i \in \{1, 2, \ldots, N\}$. The following OLC signal is applied to the terminal $c$ in the $i$th cell of Fig. 5:

$$\Delta Y_i = \begin{cases} 0, & \text{for } 2(n - 1)T < t < (2n - 1)T \text{,} \\ K_v (\frac{1}{2n-1} \bar{V}((2n - 1)T) - v_i((2n - 1)T)), & \text{for } (2n - 1)T < t < 2nT \end{cases},$$

where

$$K_v \equiv aK, \quad Y_i \equiv H(t, \bar{V}).$$

This signal is generated as the following: the S/H samples each state $v_i$ at period center $t = (2n - 1)T$ and holds it during the second half period $(2n - 1)T < t < 2nT$. Then the sampled states and its inverse ones are applied to the mean-state generator that outputs $\bar{V}$. The mean-state $\bar{X}$ in (13) is given by $\bar{V}/V_L$. The signal $\bar{V}$ is applied to each OLC signal generator, and its output signal $\Delta Y_i$ is applied to the cell. Using (9) and $\Delta y_i = \Delta Y_i / E$, we obtain (13).
Fig. 10. Laboratory measurements of the OLCN. (a) Synchronization of chaos in $C_1$. Parameters are as in Fig. 6(a) (and in Fig. 4(a)). (b) Chaotic itinerancy in $C_3$. Parameters are as in Fig. 6(b) (and in Fig. 4(d)). (c) Synchronization of stable periodic orbits in $P_1'$. Parameters are as in Fig. 6(c) (and in Fig. 4(f)). The left side shows the measured time-domain waveforms without the OLC. $v_1$ and $v_2$ correspond to the 2-periodic attractor of $F$. $v'$ corresponds to the 1-periodic attractor of $F$. The right shows the measured time-domain waveforms with the OLC.

First, we have applied this OLC signal to three cells ($N = 3$) whose parameters are in $C_1$. In this case, we have confirmed six kinds of synchronization of chaos and one of them is shown in Fig. 10(a). Second, we have applied the OLC signal to the cells whose parameters are in $C_3$. In this case synchronization of chaos is hard to be held, however, an aperiodic switching phenomenon is observed, as shown in Fig. 10(b). It is an important phenomenon, chaotic itinerancy, defined and discussed below. Third, we prepare three cells whose parameters are in $P_1'$; the second cell exhibits a stable 1-periodic orbit and the other cells exhibit stable 2-periodic orbits as shown in Fig. 10(c). When the OLC signal is applied, the second state changes to the stable 2-periodic orbit: the OLCN exhibits synchronization of stable 2-periodic attractor and three states fall into a synchronous state, as shown in Fig. 10(c).

In order to consider the laboratory measurements, we define some phenomena.

**Definition 3** The $i$th cell and the $j$th cell are said to be synchronized if $|y_i x_i - y_j x_j| < \epsilon$ is satisfied for $2n \leq \tau$, where $\epsilon$ is a sufficiently small value. The $i$th cell and the $j$th cell are said to be synchronized temporarily if there exists a finite number $n_t \geq 2n + 1$ such that $|y_i x_i - y_j x_j| < \epsilon$ is satisfied for $2n < \tau < n_t$. The OLCN is said to exhibit synchronization (respectively, temporary synchronization) if all cells are synchronized (respectively, synchronized temporarily). The OLCN is said to exhibit chaotic itinerancy if it exhibits an aperiodic switching among plural kinds of temporary synchronization.

**Remark** Referring to [2] and [3], chaotic itinerancy implies an aperiodic switching among regions where various kinds of attractors existed: an orbit wanders among the regions. In the
above definition, synchronization and temporary synchronization correspond to the attractor and the region, respectively. As a related phenomenon to the chaotic itinerancy, [18] has introduced on–off intermittency, an aperiodic switching between static behavior and chaotic burst; however, it is hard to give general definitions for chaotic itinerancy and on–off intermittency.

If the chaotic attractor \( I_{0} \) does not include the discontinuous point \( z = 1 \), repeating the OLC can hold the synchronization of chaos. In this case, the OLCN can exhibit \( 2^N \) kinds of synchronization of chaos and we can observe any of them depending on the initial state. However, if the chaotic attractor includes the discontinuous point \( z = 1 \), holding the synchronization is hard. The discontinuous point corresponds to the edge point \( (x, y, \tau) = (1, -1, 2n - 1) \) or its symmetric one (see Fig. 11). Each chaotic orbit must pass through a neighborhood of the edge; orbits of some cells hit the threshold and their outputs change but orbits of the other cells do not hit the threshold. Hence, the temporary synchronization is destroyed by small noise. Then, just after the destruction, the OLC signal is applied again and the system recovers a temporary synchronous state of chaos. Repeating in this manner, the system exhibits chaotic itinerancy. Recalling Theorem 3, \( F \) exhibits an \( m \)-periodic attractor for \((\lambda, p) \in P_m\); if we apply the OLC to such cells, the OLCN can exhibit \( 2^N \times 2 \) kinds of synchronization of stable periodic orbits. We can observe any of them depending on the initial state. Similar consideration is possible for \((\lambda, p) \in P_m\).

In conclusion, we can summarize the results as the following.

- If \((\lambda, p) \in C_3\) then the OLCN exhibits chaotic itinerancy.
- If \((\lambda, p) \in P_m\) then the OLCN will exhibit \( 2^N \) kinds of synchronization of stable \( m \)-periodic orbits. We can observe any of them depending on the initial state.
- If \((\lambda, p) \in P_m\) then the OLCN will exhibit \( 2^N \times 2 \) kinds of synchronization of stable \( m \)-periodic orbits or that of stable \( (m + 1) \)-periodic orbits. We can observe any of them depending on the initial state.

However, in the case of \((\lambda, p) \in C_2\), it is still not clear whether the OLCN exhibit synchronization of chaos or chaotic itinerancy. In our experiments, chaotic itinerancy seems to be dominant.

In the bifurcation diagram (Fig. 7), we note that synchronization of chaos changes to chaotic itinerancy as time constant parameter \( \lambda \) decreases in the chaotic phase of \((p > 0)\); but such itinerancy is impossible in the periodic phase of \((p < 0)\). The chaotic itinerancy may relate to memory search in artificial neural networks as suggested in [4]; hence the OLCN and its bifurcation may be developed into an effective associate memory.

V. CONCLUSION

We have considered the OLCN, a simple network using the occasional linear connection. The cell includes bipolar hysteresis whose thresholds vary periodically and exhibits various stable periodic orbits and chaos. The related bifurcation diagram is given theoretically. The OLCN exhibits various kinds of synchronization of stable periodic orbits and that of chaos. The OLCN also exhibits chaotic itinerancy. Introducing the concept of temporary synchronization, we define the chaotic itinerancy and clarify its generation mechanism, where discontinuity and coexistence of chaotic attractors play important roles. These results are guaranteed theoretically and verified in the laboratory. In order to develop the OLCN into more generalized networks, we are considering the following problems:
(1) detailed analysis of bifurcation phenomena from the OLCN;
(2) simple implementation of the OLCN on a chip.

**APPENDIX**

**Proof of Theorem 3** For $-1 < p < 0$ and $m \geq 2$, we assume the following (see Fig. 11): 

$$F(1_+) < 1 < F^{m-1}(1_+) < 1 < F^m(1_+). \quad (16)$$

In this case we can construct the $m$-fold composition of $F$ for $1 < z < 2$ as shown in Fig. 11: the $F^m$ is monotone decreasing and is continuous. Equation (10) guarantees $|DF^m| < 1$ for $1 < z < 2$. Since $F^m(1_+) > 1 > F^m(2_-)$, there exists a point $z_0$ such that $F^m(z_0) = z_0$; $z_0$ is stable at $m$-periodic point. If $F^m(F(1)) > F(1)$, $z_0$ is the unique fixed point of $F^m$ on $J_0 \equiv (1, F(1))$ and all orbits eventually enter into $J_0$. Also, using (3) and (5), (16) and $F^m(F(1)) \geq 1$ can be recast as $(\lambda, p) \in P_m$. That is, the first half of Theorem 3 is proven for $m \geq 2$.

If $F^m(F(1)) < 1$, there exists a point $\alpha$ in $J_0$ such that $F^m(\alpha) = 1$. Then $F^{m+1}(\alpha) = F(1) > F^{m+1}(F(1))$ is satisfied. $F^{m+1}$ is continuous, monotone decreasing with $|DF^{m+1}| < 1$ on $J_0 = [\alpha, F(1)]$. Hence there exists a unique fixed point $z_0 = F^{m+1}(z_0)$ that is an $(m+1)$-periodic point. All orbits eventually enter into $J_0$ and converge to either the $m$-periodic attractor or $(m + 1)$-periodic attractor. Also, (16) and $F^m(F(1)) < 1$ can be recast by $(\lambda, p) \in P_{m+1}$. That is, the second half of Theorem 3 is proven for $m \geq 2$.

In addition, domain of attraction for the $m$-periodic attractor (respectively, $(m + 1)$-periodic attractor) is given by $J_0 \equiv (1, \alpha)$ (respectively, $J_0$) and union of all its inverse images. The case of $m = 1$ can be proved by using analogous manner.

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