Reality of Chaos in Four-Dimensional Hysteretic Circuits

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Abstract — This paper gives a rigorous evidence for chaos in a certain class of four-dimensional hysteretic circuits. The circuit dynamics are described by two symmetric three-dimensional linear equations which are connected to each other by hysteresis switchings. We transform the circuit equation into the Jordan form and derive the two-dimensional return map $T$. Then we prove a sufficient condition for $T(D') \subset D'$ and $|DT| > 1$ on $D'$, where $D'$ is some subset in the domain of $T$ and $DT$ is the Jacobian. It implies that $T$ exhibits area-expanding chaotic attractors.

I. INTRODUCTION

In the previous paper [1], we have discussed a family of autonomous circuits that have only one nonlinear element: a piecewise-linear symmetric hysteresis resistor. This family is governed by an $M$-dimensional equation that includes one small parameter $\epsilon$. Developing the singular perturbation theory for $\epsilon \to 0$, we can simplify the equation into the constrained system: two symmetric $(M - 1)$-dimensional linear equations that are connected to each other by hysteresis switchings. Hence we can derive the $(M - 2)$ dimensional noninvertible return map. This mapping enables us to investigate various interesting phenomena in detail. Especially for the two dimensional map from four-dimensional circuits, we have observed area expanding chaos, hyperchaos [2], and related interesting phenomena. However, no mathematical evidence for chaos is given. (For some three-dimensional systems, chaos is proved. See [3]–[5] and references therein.)

This paper proves a sufficient condition for the following:

$$T(D') \subset D' \quad \text{and} \quad |DT| > 1 \quad \text{on} \quad D'$$

where $T$ is the two-dimensional return map from the four-dimensional hysteretic circuits, $D'$ is some subset in the domain of $T$, and $DT$ is the Jacobian of $T$. $|DT| > 1$ implies that the mapping $T$ expands the area, and therefore any attractor of $T$ is neither a periodic orbit nor a torus. Also, if there exist Lyapunov exponents of $T$, $\mu_1 > \mu_2$, then $\mu_1 + \mu_2$ is given by the time-average of $|\text{ln} DT|$ [6]. Therefore, (1) guarantees $\mu_1 + \mu_2 > 0$. Moreover, it is conjectured in [7] that the fractal dimension of the attractor from $T$ is 2, if $\mu_1 + \mu_2 > 0$. (Note that (1) does not guarantee hyperchaos, $\mu_2 > 0$; it is hard to prove it.) We emphasize that this paper gives the first rigorous result for area-expanding chaotic map from four-dimensional autonomous systems. At present, some works on higher dimensional chaotic systems are published [2], [8]–[14], and especially in [8] and [9], the area-expanding chaotic map is discussed. However, no mathematical evidence for chaos is given in these works. Also, the area-expanding chaotic map cannot be obtained from three-dimensional systems. In typical three-dimensional systems, e.g., Lorenz system [15], the chaotic attractor from its return map appears to be the product of one continua and a Cantor set.

We analyze the four-dimensional case as follows. In Section II, we transform the constrained system into the Jordan form and derive the two-dimensional return map. In Section III, we prove the main theorem which gives a sufficient condition for (1), by using a convenient transformation of state variables. The condition is given by some inequalities, which are rigorously described by only parameters. In Section IV, we check the inequalities by using the computer. Also, we give an example circuit. In Section V, we enumerate some remarks.

II. JORDAN FORM AND RETURN MAP

Let us begin with summarizing the simplification of the four-dimensional equation discussed in [1]:

$$
\begin{aligned}
\dot{X}_1 &= 0 \\
\dot{X}_2 &= -\alpha_0 - \alpha_1 - \alpha_2 \quad \text{and} \quad |X_1| - |X_2| (z - X_3) \\
\dot{X}_3 &= \frac{1}{\tau} X_1 - \frac{1}{\tau} X_2 - \frac{1}{\tau} X_3 - \frac{1}{\tau} X_4 - \frac{1}{\tau} X_5 - \frac{1}{\tau} X_6 \\
\epsilon \dot{z} &= X_1 - h(z)
\end{aligned}
$$

where $h(z) = z + 0.5(1 + \eta^{-1})|X_1 - X_2| - |X_1 + X_2|$ (see Fig. 1), $\frac{d}{dt}$ denotes the differentiation by $t$, $\alpha$ and $\beta$ are parameters, and $\epsilon$ is a small parameter. Fig. 2 shows a circuit family that is governed by this equation. If the linear network $N$ has two memory elements ($L$ or $C$) and has no tie-set of capacitors and no cut-set of inductors, then any circuit in this family can be systematically analyzed by using (2). Letting $\epsilon \to 0$, (2) is simplified into the
following two symmetric linear equations, which are connected to each other by hysteresis switchings:

\[
\begin{align*}
\dot{X}_1 &= \left( \begin{array}{c}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_0 & -\alpha_1 & -\alpha_2
\end{array} \right) \begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}
\text{"-" for } S_+
\end{align*}
\]

\[
\begin{align*}
\dot{X}_2 &= \left( \begin{array}{c}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_0 & -\alpha_1 & -\alpha_2
\end{array} \right) \begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}
\text{"+" for } S_-
\end{align*}
\]

where \( (p_1, p_2, p_3) = \left( \frac{-\alpha_1 \beta_1 - \alpha_2 \beta_2 - \beta_3 (1 + \eta)}{\alpha_0 \beta_1 (1 + \eta)}, \beta_2 (1 + \eta) \right) \). \( S_+ = \{ \, X, z | z - X_1 = 1 + \eta, X_1 > 1 \} \), \( S_- = \{ \, X, z | z - X_1 = -1 - \eta, X_1 < 1 \} \) \( X = (X_1, X_2, X_3) \)T, and we omit \( z \) hereafter.

Switchings: If a solution on \( S_+ \) (respectively, \( S_- \)) reaches the threshold \( X_1 = -1 \) (respectively, \( X_1 = 1 \)), then it jumps onto \( S_- \) (respectively, \( S_+ \)), holding \( X \) constant. The detail on this simplification is discussed in section II of [1].

In this paper, we concentrate our attention to the case where the coefficient matrix in (3) has real and complex eigenvalues: \(-\Lambda, \Lambda \pm j\omega\). Then, using the transformation

\[
\begin{align*}
X_1 &= 1 \\
X_2 &= \Delta \omega - \Lambda \\
X_3 &= \Delta^2 - \omega^2 \\
\end{align*}
\]

(4)

\( \text{ab. } X = S \cdot u \)

(3) can be transformed into the Jordan form:

\[
\begin{align*}
\dot{u}_1 &= \left( \begin{array}{c}
\delta & 1 & 0 \\
-1 & \delta & 0 \\
0 & 0 & -\Lambda
\end{array} \right) \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
\text{"-" for } S_+ \quad (5-1)
\end{align*}
\]

\[
\begin{align*}
\dot{u}_3 &= \left( \begin{array}{c}
\delta & 1 & 0 \\
-1 & \delta & 0 \\
0 & 0 & -\Lambda
\end{array} \right) \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
\text{"+" for } S_- \quad (5-2)
\end{align*}
\]

where \( \text{d} = d/d\tau, \delta = \Delta / \omega, \Lambda = \Lambda / \omega, \) and \( (q_1, q_2, q_3)^T = S^{-1}(p_1, p_2, p_3)^T \). Also, switchings of this equation are given by substituting \( u_1 + u_3 \) for \( X_1 \). Henceforth we use the symbol \( \tau \) instead of \( \tau' \).

For a simplicity, we restrict parameters as the following:

\[
0 < \lambda < \delta < 0.5, \quad -1 < q_3 < 0, \quad 1 < q_2, \quad q_1 - 1 < |q_3|.
\]

The solution on \( S_+ \) for \( (\tau, u_1, u_2, u_3) = (0, u_{10}, u_{20}, u_{30}) \) is given by

\[
\begin{align*}
\dot{u}_1 &= u_1 - u_3 \\
\dot{u}_2 &= u_2 - q_2 \\
\dot{u}_3 &= u_3 - q_3
\end{align*}
\]

\( \text{where } e^A(t) = \left( \begin{array}{c}
\cos \tau & \sin \tau & 0 \\
-\sin \tau & \cos \tau & 0 \\
0 & 0 & e^{-(\lambda + j\omega)t}
\end{array} \right) \left( \begin{array}{c}
u_{10} - u_1 \\
u_{20} - u_2 \\
u_{30} - u_3
\end{array} \right) \text{.} \quad (7)
\]

and the solution on \( S_- \) is symmetric to this.

Fig. 3 shows a sketch of the vector field of the Jordan form on \( S_+ \). In this figure, \( E' \) (respectively, \( E'' \)) denotes the eigenspace that corresponds to the real eigenvalue \( -\Lambda \) (respectively, complex eigenvalues \( \delta \pm j \)):

\[
E' = \{ \, u | u_1 = q_1, u_2 = q_2, u_1 + u_3 > -1 \} \quad E'' = \{ \, u | u_3 = q_3, u_1 + u_3 > -1 \}.
\]

Also, let \( P_0 \) be the boundary between the increase and the decrease of \( u_1 + u_3 \) (\( X_1 \)), \( P_+ \) be the increasing region of it, and let \( P_- \) be the decreasing region of it:

\[
P_0 = \{ \, u \mid u_1 - u_3 = 0, u_1 + u_3 > -1 \} \quad P_+ = \{ \, u \mid u_1 - u_3 > 0, u_1 + u_3 > -1 \} \quad P_- = \{ \, u \mid u_1 - u_3 < 0, u_1 + u_3 > 1 \}.
\]

Moreover, we define some other objects:

\[
B_+ = \{ \, u | u_1 + u_3 = 1, u_3 < |q_3| \} \quad B_- = \{ \, u | u_1 + u_3 = -1, u_3 > |q_3| \}
\]

\[
D_+ = B_+ \cap P_+, \quad D_- = B_- \cap P_-
\]

\( D'_- \) is the symmetric object of \( D_- \), \( L_1 = B_+ \cap P_0, \quad L_2 = B_- \cap P_0 \), \( L'_2 \) is the symmetric object of \( L_2 \).

\[
\text{(10)}
\]
Here, letting points on $D_+$ or $D_-$ be represented by their $u_1$ and $u_2$ coordinate, we consider a trajectory started from any point $(u_{10}, u_{20})$ on $D_+$ at $\tau = 0$. Since the real eigenvalue component goes toward $E^c$ and since the complex eigenvalue component divergently rotates round $E'$, the trajectory must hit $D_-$ at some positive time $\tau_1$. Let $(u_{11}, u_{21})$ be this hit point. Note that it never hit the outside of $D_-$ on $S_-$, because $u_1 + u_3$ increases on there. At this hit moment, it jumps onto the same point in $S_-$. Noting that a trajectory started from $(u_{11}, u_{21})$ on $D_- \subset S_-$ is symmetric to that started from $(-u_{11}, -u_{21})$ on $D_- ', S_-$, we can define the two-dimensional return map

$$T: D_+ \to D_-, \quad (u_{10}, u_{20}) \to (-u_{11}, -u_{21}).$$

This mapping can be calculated by

$$\begin{pmatrix} u_{11} - q_1 \\ u_{21} - q_2 \end{pmatrix} = \begin{pmatrix} f_1(\tau_1, u_{10}, u_{20}) \\ f_2(\tau_1, u_{10}, u_{20}) \end{pmatrix} = e^{\lambda \tau_1} \begin{pmatrix} \cos \tau_1 & \sin \tau_1 \\ -\sin \tau_1 & \cos \tau_1 \end{pmatrix} \begin{pmatrix} u_{10} - q_1 \\ u_{20} - q_2 \end{pmatrix}$$

where the hit time $\tau_1$ is given by

$$(u_{10} - q_1, u_{20} - q_2, 1 - u_{10} - q_3) \tau_1 + q_1 + q_3 + 1 = 0. \quad (13)$$

Here, we note that trajectories that touch $L_2$ raise discontinuities of $T$; $T$ is piecewise differentiable. This discontinuity plays an important role for the folding mechanism of chaos.

Fig. 4 shows a trajectory of the Jordan form and some examples of attractors of $T$, where (13) is solved by using the Newton Raphson method. In this figure, the attractor of (c) corresponds to the trajectory of (a) and trajectories corresponding to (b) and (d) are omitted. As $q_3$ is decreased, the form of the attractor becomes thicker.

### III. Chaos Generating Condition

In this section, we consider the Jordan form on $S_+$ (5-1) and the following linear system:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} \delta & 1 \\ -1 & \delta \end{pmatrix} \begin{pmatrix} u_1 - q_1 \\ u_2 - q_2 \end{pmatrix}.$$  \hspace{1cm} (14)

Equation (14) is the projection of (5-1), which is extended for the whole region. For convenience, we define some symbols for the projection system (see Fig. 5). Firstly, let $l_2, l_2', d_-$, and $d'_-$ denote the projection of $I_2, I_2', D_-$, and $D'_-$ onto the $u_1$ versus $u_2$ plane, respectively. They are given explicitly; for example,

$$l_2 = \{u_1, u_2 | (\lambda + \delta) (u_1 - q_1) + (u_2 - q_2) + \lambda (1 + q_3 + q_4) = 0, u_1 > 1 > |q_3| \}. \quad (15)$$

Next, let $a = (u_{1a}, u_{2a})$ denote the left-top corner of $d_-$, $b = (u_{1b}, u_{2b})$ denote the right-bottom corner of $d_- '$ and let $c = (u_{1c}, u_{2c})$ denote the left-bottom corner of $d'_-$. Then, let $U_a$ (respectively, $U_b$) denote the $u_2$-coordinate of the intersection of a half-line $l_0 = \{u_1 = q_1, u_2 < q_2 \}$ and the trajectory started from $a$ (respectively, $b$). Also, let $U_c$ denote the $u_2$-coordinate of the point on $l_0$ such that the trajectory started from $(u_1, U_c)$ passes through $c$. They are given explicitly; for example,

$$U_b = -\sqrt{(u_{1b} - q_1)^2 + (u_{2b} - q_2)^2} e^{\delta \beta} + q_2.$$
where
\[
(u_{1b}, u_{2b}) = (1 - q_3, 2\lambda q_3 - \delta(1 + q_1 + q_2) q_2)
\]
\[
\phi_b = \tan^{-1} \left( \frac{(u_{1b} - q_1)}{(q_2 - u_{2b})} \right),
\]
\[0 < \phi_b < \pi/2. (16)\]

Letting \(U^*\) denote the minimum value of \(U_u, U_v,\) and \(U_c\), we consider the trajectory started from \((q_1, U^*)\) at \(T = 0\) (Fig. 5 shows the case of \(U^* = U_b\)). Since \(U^* < U_a\), this trajectory must intersect the line \(l_a = \{u_1 = -1 + q_3\}\) at some positive time \(\tau_d\) such that \(0 < \tau_d < \pi/2 + \tan^{-1} \delta\) (note that \(u_1\) decreases by \(\tau = \pi/2 + \tan^{-1} \delta\)). Let \(d = (-1 - q_3, u_{2d})\) denote this intersection. \(\tau_d\) is uniquely given by the following equation:
\[
g(\tau_d) = -(U^* - q_2) e^{\delta \tau_d} \sin \tau_d - 1 - q_3 - q_1 = 0,
\]
\[0 < \tau_d < \pi/2 + \tan^{-1} \delta \quad (17a)\]
and \(u_{2d}\) is given by substituting this \(\tau_d\) into
\[
u_{2d} = (U^* - q_2) e^{\delta \tau_d} \cos \tau_d + q_3. \quad (17b)\]

\(u_{2d}\) and \(\tau_d\) play an important role in following parts. Here, \(d_T\) denote the parallelogram surrounded by \(d_a\) and the line through \(d\) with slope equal to that of \(l_2\) and let \(l_a\) denote the bottom side of \(d_T\). Also, let \(d_T^0\) (respectively, \(d_T^1\)) denote the symmetric object of \(d_T\) (respectively, \(l_a\)). Moreover, let \(D_T\) (respectively, \(D_T^0\)) denote the back-projection of \(d_T\) (respectively, \(d_T^0\)) to \(S_+\). Then we have the following.

**Theorem 1:** If
\[
u_{2d} > -q_2 \quad \text{and} \quad (1 - \delta(\lambda + \delta))(1 + q_3 + q_1) + (\lambda + 2\delta)(u_{2d} - q_2) > 0 \quad (18a)
\]
then \(T(D_T^0) \subset D_T^0\) is satisfied.

**Proof** (see Fig. 5): Firstly, let \(m_1\) (respectively, \(m_2\)) denote the boundary between the increase and the decrease of \(u_1\) (respectively, \(u_2\)) on projection system:
\[
m_1 = \{u_1, u_2 | \delta(u_1 - q_1) + (u_2 - q_2) = 0\}
\]
\[
m_2 = \{u_1, u_2 | -(u_1 - q_1) + \delta(u_2 - q_2) = 0\}. \quad (19a)
\]

Next, let \(m_3\) (respectively, \(m_1\)) denote the intersection of the trajectory started from \(d\) and the line \(l_K = \{u_1 = -1 + q_3\}\) for positive time (respectively, negative time) and let \(m_2\) denote the intersection of this trajectory and \(m_1\).

Then, let \(d_K\) denote the region surrounded by the arc \(m_1, m_2\) and the line \(l_K\). Here, note that \(a, b,\) and \(c\) are not located in the outside of \(d_K\). This is because \(U^*\) is the minimum value of \(U_u, U_v,\) and \(U_c\) and because both \(u_1\) and \(u_2\) decrease at \(b\).

If \(u_{2d} > -q_2\), then \(l_{a}'\) is located under \(m_1\) and therefore, \(d_T^0\) is included in \(d_K\). Consequently, any trajectory of the projection system started from \(d_T^0\) must intersect the segment \(m_1, m_2\) on \(l_K\) by the time when it goes out of \(d_K\).

Hence it follows that any trajectory of the Jordan form started from \(D_T^0\) must hit \(D_T^0\) if the arc \(d_3\) is not located under \(l_a\).

On the other hand, let \(m_3\) denote the boundary between the increase and the decrease of \((\lambda + \delta)u_1 + u_2:
\[
m_2 = \{u_1, u_2 | (\delta(\lambda + \delta) - 1)(u_1 - q_1) + (\lambda + 2\delta)(u_2 - q_2) = 0\}. \quad (19b)
\]

The left-hand side of the condition (18b) is given by substituting \((-1 - q_3, u_{2d})\) into (19b), and therefore, (18b) implies that \(d\) is located in the left-side of \(m_3\). That is, the arc \(d_3\) of the projection system is not located under \(l_a\). QED

For the main theorem, we restrict the domain of \(T\) in \(D_T^0:\n\]
\[
T: D_T^0 \rightarrow D_T, \quad (u_{10}, u_{20}) \rightarrow -(u_{11}, u_{21}). \quad (11')
\]

Next, we give a convenient formulation for the Jacobian of \(T\).

**Theorem 2:**
\[
|dT| = e^{(2\delta - \lambda)\tau_1} \frac{(\lambda + \delta)(u_{10} - q_1) + (u_{20} - q_2) - \lambda(1 - q_3 - q_1)}{(\lambda + \delta)(u_{11} - q_1) + (u_{21} - q_2) + \lambda(1 + q_3 + q_1)} \quad (20)
\]

where \(dT\) is the Jacobian of \(T\) and the hit time \(\tau_1\) is given by (13).

**Proof:** The Jacobian matrix \(dT\) is given by using \(f_1, f_2,\) and \(f_3\) in (12) and (13):
\[
dT = \frac{\partial(u_{11}, u_{21})}{\partial(u_{10}, u_{20})} \frac{\partial(f_1, f_2)}{\partial(u_{10}, u_{20})} - \frac{1}{f_3^2} \left[ f_1 \frac{\partial f_3}{\partial u_{10}} + f_2 \frac{\partial f_3}{\partial u_{20}} \right] \quad (21)
\]

where \(\cdots\) denote the differentiation by \(\tau_1\). Substituting the right-hand sides of (12) and (13) into (21) and calcu-
Substituting (23a) into (22), we obtain (20).

Moreover, we note that \( f_3' \) (respectively, \( G(\cdot) \)) is equal to the differentiation of \( u_1 + u_3 \) by \( \tau \) at a point \((u_{10},u_{20}) \in D_T \) (respectively, a point \((u_{10},u_{20}) \in D_T \)) and therefore,

\[
f_3' < 0, \quad G(\cdot) < 0 \quad \text{(23b)}
\]
is satisfied. That is, both the numerator and the denominator of (20) are negative.

Now we turn to discuss the main theorem. First, we define \( \theta_1 \) and \( \theta_2 \) in the triangle given by \( l_2 \) and the equilibrium point \( q \) (see Fig. 6):

\[
\theta_1 = \tan^{-1} \frac{q_1 - 1 - q_2}{q_2 + u_{2d}}, \quad \left( 0 < \theta_1 < \frac{\pi}{2} \right)
\]

\[
\theta_2 = \tan^{-1} \frac{1 - q_3 - q_1}{q_2 + u_{2d} - 7q_3(\lambda + \delta)}, \quad \left( 0 < \theta_2 < \frac{\pi}{2} - \varphi_n \right)
\]

\[
\varphi_n = \tan^{-1} (\lambda + \delta).
\]

Next, we introduce the following important transformation:

\[
\begin{bmatrix}
 u_{10} - q_1 \\
 u_{20} - q_2
\end{bmatrix}
 = e^{\delta \tau_\alpha}
\begin{bmatrix}
 \cos \tau_\alpha & \sin \tau_\alpha \\
 -\sin \tau_\alpha & \cos \tau_\alpha
\end{bmatrix}
\begin{bmatrix}
 0 \\
 u_\alpha - q_2
\end{bmatrix}.
\quad \text{(25a)}
\]

This formula implies that a trajectory of the projection system started from \((q_1,u_\alpha)\) at \( \tau = 0 \) reaches \((u_{10},u_{20}) \) at \( \tau = \tau_\alpha \). It defines the transformation between \((u_{10},u_{20})\) and \((u_\alpha,\tau_\alpha)\), where \( u_\alpha \) and \( \tau_\alpha \) satisfy

\[
\begin{align*}
 U^* < u_\alpha < q_2, \\
 -\theta_2 < \tau_\alpha < \theta_1.
\end{align*}
\quad \text{(25b)}
\]

By using the analogous manner, the transformation between \((u_{11},u_{21})\) and \((u_\alpha,\tau_\alpha)\) is given by

\[
\begin{bmatrix}
 u_{11} - q_1 \\
 u_{21} - q_2
\end{bmatrix}
 = e^{\delta (\tau_\alpha + r_\alpha)}
\begin{bmatrix}
 \cos (\tau_\alpha + r_\alpha) & \sin (\tau_\alpha + r_\alpha) \\
 -\sin (\tau_\alpha + r_\alpha) & \cos (\tau_\alpha + r_\alpha)
\end{bmatrix}
\begin{bmatrix}
 0 \\
 u_\alpha - q_2
\end{bmatrix}.
\quad \text{(26a)}
\]

Here, let a trajectory of the projection system started from a point \((q_1,u_\alpha) \in D_T \) reach \((u_{11},u_{21}) \) after \( n \)-times intersections with \( I_0 \). If \( n = 0 \), the trajectory does not enter into \( d_T \) by \( \tau = \tau_\alpha \) and it must go out of \( d_T \) by \( \tau = \pi/2 + \varphi_n \) (see Fig. 6). Therefore, the following is satisfied:

\[
2n\pi + \tau_d < \tau_\alpha + \tau_\alpha < 2n\pi + \pi/2 + \varphi_n.
\quad \text{(26b)}
\]

We note that \( n \) is finite, because the equilibrium point \( q \) is located in the outside of \( d_T \). Then we obtain the main theorem.

**Theorem 3:** If

\[
- q_2 < u_{2d} < - (1 + q_3) q_2 / q_1
\]

\[
(1 - \delta(\lambda + \delta))(1 + q_3 + q_1) + (\lambda + 2\delta)(u_{2d} - q_2) > 0
\]

\[
\tau_d - \tau_2 - 2\varphi_n > 0
\]

\[
e^{\delta - \lambda(\tau_d - \theta_1 + 2\varphi)} \frac{\cos (\theta_1 - \varphi_n)}{\cos (\tau_d - \varphi_n)} - 1 > 0
\]

then the following is satisfied:

\[
T(D_T) \subset D_T, \quad \text{\( |DT| > 1 \) on \( D_T \).} \quad \text{(1)}
\]

Proof. First, we note that (18b) and the left part of (27a) are the conditions in Theorem 1. Also, the right-hand side of (27a) denotes the \( u_\alpha \)-coordinate of \( c \) in Fig. 6, which is the intersection of \( l_2 \) and the line through \( q \) and the origin. It guarantees

\[
\tau_d < \theta_1 < \pi/2.
\quad \text{(28)}
\]

Next, substituting (25a) and (26a) into (20), we obtain

\[
|DT| = e^{c2\delta - \lambda r_\alpha}
\]

\[
A_m e^{\delta \tau_\alpha} \cos (\tau_\alpha - \varphi_n) - \lambda(1 - q_3 - q_1)
\]

\[
A_m e^{\delta (\tau_\alpha + r_\alpha)} \cos (\tau_\alpha + \tau_\alpha - \varphi_n) + \lambda(1 + q_3 + q_1)
\]

where \( A_m = \sqrt{(\lambda + \delta)^2 + 1} \ (u_\alpha - q_2) \).

Here, noting (25b) and (26b), we notice

\[
|\tau_\alpha - \varphi_n| < \pi/2
\]

\[
2n\pi - \pi/2 < \tau_\alpha + \tau_\alpha - \varphi_n < 2n\pi + \pi/2.
\quad \text{(30a)}
\]

Also, the following is assumed in (6) and (25b):

\[
\delta > \lambda, \quad \lambda(1 - q_3 - q_1) > 0
\]

\[
(1 + q_3 + q_1) > 0, \quad u_\alpha - q_2 < 0.
\quad \text{(30b)}
\]

In addition, (23b) implies that both the numerator and the denominator of (29) are negative. Noting these facts,
we can obtain the following:

\[ |DT| > e^{(\theta - \lambda)\pi} \left( \cos \left( \tau_a - \varphi_n \right) / \cos \left( \tau_a + \tau_1 - \varphi_n \right) \right). \tag{31} \]

Hereafter, we estimate this formula. First, we assume that (27b) is satisfied for \( n = 0 \). Noting (25b) and (26b), we notice that one of the following is satisfied:

\[-\pi / 2 < -\theta_2 - \varphi_n < \tau_a - \varphi_n < 0 \quad \text{and} \]
\[ \theta_2 + \varphi_n < \tau_a + \tau_1 - \varphi_n < \pi / 2 \quad \text{(32a)} \]
\[ 0 < \tau_a - \varphi_n < \tau_a + \tau_1 - \varphi_n < \pi / 2. \quad \text{(32b)} \]

This fact implies that the right-hand side of (31) is greater than 1.

Next, we assume that (27b) and (28) are satisfied for \( n \neq 0 \). Noting (25b) and (26b), we obtain

\[ 2n\pi + \tau_d - \theta_1 < \tau_1 \quad \text{(33a)} \]

and therefore,

\[ e^{(\theta - \lambda)\pi} > e^{(\theta - \lambda)\tau_d - \theta_1 + 2\pi}. \quad \text{(33b)} \]

Also, noting (25b), (27b), and (28), we obtain

\[-(\theta_2 + \varphi_n) < \tau_a - \varphi_n < \theta_1 - \varphi_n < \pi / 2 \quad \text{(34a)} \]
\[ \theta_2 + \varphi_n < \theta_1 - \varphi_n. \quad \text{(34b)} \]

It guarantees

\[ \cos \left( \tau_a - \varphi_n \right) > \cos \left( \theta_1 - \varphi_n \right) > 0. \quad \text{(34c)} \]

Moreover, noting (26b), (27b), and (28), we obtain

\[ 2n\pi < \tau_d + 2n\pi - \varphi_n < \tau_a + \tau_1 - \varphi_n < 2n\pi + \pi / 2. \quad \text{(35a)} \]

It guarantees

\[ 0 < \cos \left( \tau_a + \tau_1 - \varphi_n \right) < \cos \left( \tau_a - \varphi_n \right). \quad \text{(35b)} \]

Finally, noting (31), (33b), (34c), and (35b), we obtain

\[ |DT| > e^{(\theta - \lambda)\pi} \left( \cos \left( \theta_1 - \varphi_n \right) / \cos \left( \tau_a - \varphi_n \right) \right). \quad \text{(36)} \]

Letting the right-hand side be greater than 1, we obtain (27c).

Note that the sufficient condition is described by only parameters \((\lambda, \delta, q_1, q_2, q_3)\).

IV. VERIFICATION OF THEOREM 3

If \( q_3 \) is selected as a control parameter, each term in the condition of Theorem 3 is calculated as shown in Fig. 7. This figure indicates that \( U^* = U_a \) for \( q_3 < q_a \), \( U^* = U_b \) for \( q_3 > q_a \), (27a) is satisfied for \( q_3 < q_a \), and both (27b) and (27c) are satisfied for \( q_3 > q_a \). In addition we have verified that (18b) is satisfied for the interval of \( q_3 \) in this figure. Consequently, all conditions in Theorem 3 are satisfied for \( q_a < q_3 < q_b \). Fig. 8 shows the region in which all conditions are satisfied. Each curve is calculated by using the regula falsi. Also, attractors in Fig. 4(b), (c), and (d) are obtained at each “x” in this figure. We note that \( |DT| > 1 \) is numerically verified even for the attractor at \( q_3 = -0.4 \) by \( 10^5 \) times iterations. It goes without saying that the analogous manner is valid for all other parameter pairs.

In order to show laboratory data, we propose an example circuit of Fig. 9. In this figure, \(- r_a \) is a linear negative resistor and \( NR \) is a nonlinear resistor characterized by Fig. 2(b). The realization of \(- r_a \) and \( NR \) is shown in [1].
We define dimensionless variables and parameters:
\[
\tau = \frac{t}{r_1 C}, \quad x_1 = \frac{v_1}{V}, \quad x_2 = \frac{r_1 i}{V}, \quad x_3 = \frac{v_2}{V},
\]
\[
\gamma = \frac{C_1}{C_2}, \quad \rho = \frac{r_1^2 C_1}{L}, \quad \sigma_1 = \frac{r_2}{r_1},
\]
\[
\sigma_2 = \frac{r_1}{r_b}, \quad \eta = \frac{r_1}{r_2}
\]
(37)
where \( V, r_1, \) and \( r_2 \) are shown in Fig. 2(b). By using these, parameters and variables of the switching system (3) are given by
\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
-1 - \sigma_2 & 0 & 0 \\
(1 + \sigma_2)^2 - \rho + \gamma \sigma_2^2 & 1 + \sigma_2 - \rho \sigma_1 & -\sigma_2(1 + \sigma_2 + \gamma \sigma_2)
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]
(38)
\[
\alpha_0 = \rho \gamma \sigma_2(1 - \sigma_1)
\]
\[
\alpha_1 = \gamma \sigma_2(1 - \rho \sigma_1) + \rho(1 - \sigma_1 - \sigma_1 \sigma_2)
\]
\[
\alpha_2 = 1 - \rho \sigma_1 + \sigma_2(1 + \gamma)
\]
\[
p_1 = \frac{\sigma_1}{1 - \sigma_1}(1 + \eta), \quad p_2 = 1 + \eta,
\]
(39)
Here, we note that the number of parameters in (37) is the same as that in the Jordan form (5). This circuit exhibits chaotic trajectory as shown in Fig. 10(a) which corresponds to the trajectory of Fig. 4(a). Fig. 10(b) shows the transformed trajectory of Fig. 4(a).

V. Concluding Remarks
For a certain class of four-dimensional hysteretic circuits, we have shown the rigorous result for area-expanding chaotic return map. This is the first result for four-dimensional autonomous systems. Finally, we enumerate some remarks.

1) The condition is described by only parameters. It is verified by the computer, but the mathematical verification seems to be hard.

2) The condition should be extended for more wide range of parameters. For this purpose, a computer-assisted proof [3] seems to play an important role.

3) Only one example circuit is given. However, various other examples exist and they seem to exhibit various interesting phenomena. They will be summarized in the future.

Also, various hard problems remain; for example:

4) a chaos generating condition for nonconstrained systems;

5) existence of Lyapunov exponent and estimation of fractal dimension; and

6) classification of chaotic attractors.

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