Abstract — Two types of windows are found in a symmetric circuit, which cannot be seen in the logistic map and are inherent in symmetric structure of this circuit. One type of window exhibits complex bifurcation phenomena such as the symmetry breaking and the symmetry recovering in its own region, while the other type of window appears when one chaotic attractor bifurcates to two distinct periodic attractors. The purpose of this paper is to derive a one-dimensional Poincaré map from the circuit by the degeneration technique and to prove rigorously that two types of windows appear alternately and infinitely many times inside some windows by applying a certain scaling mechanism on the Poincaré map.

I. INTRODUCTION

SYMMETRIC structures are often observed in the chaos-generating circuits [1]-[8]. For example the V-I characteristic of nonlinear resistance in the circuit discussed in [2]-[4] is symmetric with respect to origin. The circuit equations of these systems are invariant to the transformation \((x, y, z) \rightarrow (-x, -y, -z)\). The “symmetry” is one of the features of the chaos-generating circuits. Therefore, it is of great importance to investigate the bifurcation phenomena based on the symmetry of circuits, such as the symmetry breaking, the symmetry-recovering [3], and so on.

In the meantime, some symmetric one-dimensional maps have been studied [9]-[11]. For example, Testa et al. [11] investigated the cubic map and discovered two types of the windows. May [9] has introduced the term “window” for the interval in parameter space in which the periodic solution remains stable [12]. It is called “window” since the area of its existence on a one-parameter bifurcation diagram is placed in the area of chaos like windows. The two types of the windows discovered by Testa et al. [11] cannot be seen in the logistic map, namely, one exhibits complex bifurcation phenomena such as the symmetry breaking and the symmetry recovering in its own region, while the other is the windows appearing when one chaotic attractor bifurcates to two distinct periodic attractors [11]. These windows have interesting bifurcation structures and seem to be deeply related to bifurcation phenomena in symmetric real physical systems. However, there are few discussions to fill in the gap between the windows in mathematical models and those of real physical systems.

In this paper we analyze two types of the windows in a symmetric circuit described by a differential equation in a rigorous way. These windows are similar to those appearing in the cubic map. For understanding our study easily, at first we introduce some experimental and numerical results.

Let us consider the circuit in Fig. 1 whose structure is symmetric with respect to the origin. This circuit consists of three memory elements, one linear negative resistance, and one nonlinear resistance consisting of two diodes. The negative linear resistance is realized by using the linear region of the negative impedance converter made of an operational amplifier. The measured \(I-V\) characteristic of the nonlinear resistance is given in Fig. 2(a). We approximate the \(I-V\) characteristic by the following 3-segment piecewise-linear function as shown in Fig. 2(b):

\[
v_d(i_2) = \frac{r_d}{2} \left(\frac{i_2 + V}{r_d} - \frac{i_2 - V}{r_d}\right).
\]

Then the circuit dynamics is described by the following piecewise-linear third-order ordinary differential equation:

\[
\begin{align*}
L_1 \frac{d i_1}{dt} &= -v + r_1 i_1, \\
L_2 \frac{d i_2}{dt} &= v - v_d(i_2), \\
C \frac{dv}{dt} &= -i_1 - i_2.
\end{align*}
\]

By changing the variables such that

\[
i_1 = \sqrt{\frac{C}{L_1}} V x; \quad i_2 = \frac{\sqrt{L_2 C \tau}}{L_1} V y; \quad v = V z; \quad t = \sqrt{L_1 C \tau}
\]

\[
r \sqrt{\frac{C}{L_1}} = \alpha; \quad \frac{L_1}{L_2} = \beta; \quad \frac{\sqrt{L_1 C \tau}}{L_2} = \gamma; \quad \frac{d}{d \tau} = \frac{d}{dt}
\]

(2) is normalized as

\[
\begin{align*}
\dot{x} &= ax + z, \\
\dot{y} &= z - f(y), \\
z &= -x - \beta y.
\end{align*}
\]
where

\[
f(y) = \frac{\gamma}{2} \left| y + \frac{1}{y} - \frac{y - 1}{y} \right|.
\]  

Fig. 3 presents experimental results and Fig. 4 presents the associated computer simulation of (4). All parameter values are denoted in each figure where parameter \( r \) or \( \alpha \) is varied. The resistance of the diode when it is off is denoted by \( r_d \), and it is determined by measuring the \( I-V \) characteristic and applying the method of least squares [7]. Since (4) is piecewise-linear, the general solution in each region can be obtained. The times when the solution crosses the boundaries are obtained by solving the implicit equations numerically. There exists small amount of errors between the experimental results and numerical ones. It seems to be due to the stray capacitance of the diodes. These results agree well at least qualitatively. Equation (4) is justifiable in discussing the circuit in Fig. 1.

At the parameter value in figures (b)-(e) and (f) of Figs. 3 and 4, there exist two distinct attractors whose flows have different initial conditions. Because of the symmetry of the system, these two attractors locate symmetrically with respect to the origin. The circuit exhibits a variety of interesting phenomena such as the chaotic phenomenon, the symmetry-breaking transition phenomenon [13], the symmetry-recovering crises [13] and some windows [12]. Especially, the difference between the shape of the windows in Fig. 4(g) and (i) draws our attention. The periodic window in Fig. 4(g) is symmetric, while the periodic window in Fig. 4(i) is asymmetric. We call the window in Fig. 4(g) the window of Type 1 and also the window in Fig. 4(i) the window of Type 2. These two windows seem to have considerably different natures. In order to investigate detailed structures of these two types of the windows, we make the one-parameter bifurcation diagrams from calculating (4). Fig. 5(a) presents the window of Type 1 which associates with Fig. 4(g). Fig. 5(b) presents the window of Type 2 which associates with Fig. 4(i). In Fig. 5, for the parts (a1) and (b1), we give an initial condition \((x, y, z) = (-1, 0, 0)\) for (4) and plot the \(x\)-coordinate of the cross section when the flow intersects the plane of \( z = 0.2 \).

Next we increase the parameter \( \alpha \) and choose the point at which the calculation at the previous parameter value ends as a new initial condition. For the parts (a2) and (b2), we give an initial condition \((x, y, z) = (1, 0, 0)\) and perform the similar calculation. The difference between these figures means that two distinct attractors coexist. In order to characterize these windows, some symbols are introduced:

- \( P_1 \): state where one periodic attractor exists;
- \( P_2 \): state where two distinct periodic attractors exist;
- \( C_1 \): state where one chaotic attractor exists;
- \( C_2 \): state where two distinct chaotic attractors exist.

By using above symbols, these windows are represented as follows (see Fig. 5).

The window of Type 1:

\[
C_1 \rightarrow P_1 \rightarrow P_2 \overset{\text{Period-Doubling}}{\rightarrow} C_2 \rightarrow C_1 \rightarrow C_1
\]  

Window Generating Region

and the window of Type 2:

\[
C_1 \rightarrow P_2 \overset{\text{Period-Doubling}}{\rightarrow} C_2 \rightarrow C_1
\]  

Window Generating Region

These windows are characterized as interesting phenomena where the number (one or two) and the nature (periodic or chaotic) of the attractor change in a complicated manner, which are considered to be inherent in the symmetric structure of the system.

The purpose of this paper is to analyze these windows rigorously by using a degeneration technique shown in [7], [14]–[17]. This degeneration means that the diode in the circuit is assumed to operate as an ideal limiter. In this case, we can derive a one-dimensional Poincaré map described implicitly by using the general solution of the piecewise-linear differential equation and it well explains the above-mentioned phenomena. Especially, it is proved rigorously, that two types of the windows appear alternately and infinitely many times inside some windows under some reasonable assumptions by utilizing a certain scaling mechanism on the Poincaré map. That is, it is clarified that the number and the nature of the attractor change in a complicated manner. In addition, we confirmed that the generation of windows is concerned with a certain universal constant [18], [19] from the numerical calculation.

We also confirmed that the two types of the windows appear in some simple symmetric systems, for example, double scroll equation [2], Sparrow’s equation [20], and so on. It seems that the two types of windows are one of the typical behaviors which characterize symmetric chaos-generating systems.
II. IDEAL MODEL AND GEOMETRIC STRUCTURE OF THE VECTOR FIELDS

Though the mapping method is usually used for the analysis of chaos, it is very difficult to prove the generation of chaos in this case because the system to be analyzed is a two-dimensional discrete dynamical system and the mathematical theories for two- or higher dimensional discrete dynamical systems are not sufficiently developed. Therefore, in order to investigate the phenomena observed from our circuit, we must use the degeneration technique shown in [7], [14]–[17], which means that the nonlinear resistance in the circuit operates as an ideal limiter as shown in Fig. 6(a). An equivalent circuit of the nonlinear resistance is shown in Fig. 6(b). We have already verified mathematically that if the operation of the diode is sufficiently sharp, the behavior of the solution of a circuit family including a diode can be explained by idealizing the diode [14], [16], [17]. The idealized equation is given by the following.

**Ideal Model**

1. When \( i_2 > 0 \) \( (v_2(i_2) = V) \) is satisfied:

\[
L_1 \frac{di_1}{dt} = ri_1 + v
\]

\[
L_2 \frac{di_2}{dt} = v - V
\]

\[
C \frac{dv}{dt} = -i_1 - i_2.
\]
(2) When \( i_2 = 0 \) (\(|v_d(i_2)| < V\)) is satisfied:

\[
\frac{dL_1}{dt} = r_1 + v
\]

\[
\frac{dv}{dt} = -i_1.
\]

(3) When \( i_2 < 0 \) (\(v_d(i_2) = -V\)) is satisfied:

\[
\frac{dL_1}{dt} = r_1 + v
\]

\[
\frac{di_2}{dt} = v + V
\]

\[
C \frac{dv}{dt} = -i_1 - i_2.
\]

These equations in the regions (1), (2), and (3) are connected by the following transitional conditions:

1 \( \rightarrow \) 2: when the current \( i_2 \) (through the nonlinear resistance) decreases and becomes 0;

2 \( \rightarrow \) 1: when the voltage \( v \) (across the nonlinear resistance) increases and reaches a constant \( V \);

3 \( \rightarrow \) 2: when the voltage \( v \) (across the nonlinear resistance) decreases and reaches a constant \(-V\);

3 \( \rightarrow \) 2: when the current \( i_2 \) (through the nonlinear resistance) increases and becomes 0.

By changing the variables in (3), the Ideal Model (8) is normalized as the following.

\[
\frac{dL_1}{dt} = r_1 + v
\]

\[
\frac{di_2}{dt} = v + V
\]

\[
C \frac{dv}{dt} = -i_1 - i_2.
\]
Fig. 5. One-parameter bifurcation diagrams obtained by calculating (4). (Subscripts 1 and 2 denote different initial conditions.) (a) The window of Type 1. (b) The window of Type 2.

Fig. 6. Idealization of the nonlinear resistance in Fig. 1. (a) Idealized I-V characteristic. (b) Equivalent circuit.

Fig. 7. Simulated results obtained by using the ideal model (10). $L_1 = 600$ mH, $L_2 = 200$ mH, $C = 0.0069$ µF ($R = 3.0$) (a) $\alpha = 0.150$. (b) $\alpha = 0.275$. (c) $\alpha = 0.300$. (d) $\alpha = 0.308$. (e) $\alpha = 0.350$. (f) $\alpha = 0.400$. (g) $\alpha = 0.412$. (h) $\alpha = 0.460$. (i) $\alpha = 0.471$. (j) $\alpha = 0.490$. 
When $y > 0$

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\alpha & 0 & 1 \\
0 & 0 & 1 \\
-1 & -\beta & 0
\end{bmatrix}
\begin{bmatrix}
x + 1/\alpha \\
y - 1/\alpha \beta \\
z - 1
\end{bmatrix}
\]

When $y = 0$

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\alpha & 0 & 1 \\
0 & 0 & 1 \\
-1 & -\beta & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y + 1/\alpha \beta \\
z + 1
\end{bmatrix}
\]

When $y < 0$

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\alpha & 0 & 1 \\
0 & 0 & 1 \\
-1 & -\beta & 0
\end{bmatrix}
\begin{bmatrix}
x - 1/\alpha \\
y + 1/\alpha \beta \\
z + 1
\end{bmatrix}
\]

The transitional conditions are also represented by

1. $\rightarrow 2$: $y = 0$
2. $\rightarrow 1$: $z = 1$
3. $\rightarrow 3$: $z = -1$
4. $\rightarrow 2$: $y = 0$

Fig. 7 presents computer simulation results of (10). These results agree with those of (4) both qualitatively and quantitatively. This agreement shows that this method using the Ideal Model is justifiable to analyze the circuit in Fig. 1.

In the following discussions we analyze the circuit in Fig. 1 by using the Ideal Model of (10).

This equation contains two parameters, $\alpha$ and $\beta$, and is symmetric, that is, (10) is invariant with respect to the transformation $(x, y, z) \rightarrow (-x, -y, -z)$. Define:

$D^+ = \{(x, y, z): y > 0\}$

$D^o = \{(x, y, z): y = 0\}$

$D^- = \{(x, y, z): y < 0\}$

The subspaces $D^+, D^o,$ and $D^-$ represent the regions $1, 2,$ and $3$ in (10), respectively. Equation (10) is piecewise-linear and it has a unique equilibrium in each region. Let $P^+$ be the equilibrium in $D^+$ and also let $O$ and $P^-$ be the equilibria in $D^o$ and $D^-$, respectively. The equilibria are explicitly given by

$P^\pm = (\pm x_p, \pm y_p, \pm 1) = \left(\pm \frac{1}{\alpha}, \pm \frac{1}{\alpha \beta}, \pm 1\right) \subset D^\pm$

$O = (0, 0, 0) \subset D^o$

We define the following subspaces.

$B^+ = \{(x, y, z): z = \pm 1\}$

$C^o = \{(x, y, z): D^o \cap E^2(P^\pm)\}$

The subspace $B^+$ means the transitional condition $2 \rightarrow 1$ while $B^-$ means the transitional condition $2 \rightarrow 3$.

Fig. 8 shows the structure of the vector field. Since (10) is symmetric, we show only the regions $D^+$ and $D^o$. We describe the flow of the solution for the initial condition on $D^o$ near $O$. The solution rotates divergently.
where $r_A$ is given by the following implicit equation:

$$Z_m = [0,1] \times F_2(0) \times F_2^{-1}(r_A) \times \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad 0 < \omega_0 r_A < \pi. \quad (21)$$

We consider the solution whose initial condition is represented by (22), namely the solution having the initial condition on $L^+$:

$$\begin{pmatrix} \tau, x, y, z \end{pmatrix} = (0, x_0, 0, -Z_m). \quad (22)$$

If the parameter values are chosen properly, the solution which leaves the line $L^+$ hits $L^+$ again at some time.

1) $0 < x_0 \leq A$: The solution rotates divergently around $O$ constrained onto $D^o$ and it hits $L^-$ at the time $\tau = \tau_0$ without reaching the transitional condition $x = -1$.

2) $A < x_0$: The solution hits the line $B^-$ at $(\tau, x, y, z) = (\tau_1, x_1, 0, -1)$ and enters $D^-$. The solution which entered $D^-$ hits the plane $D^o$ at $(\tau, x, y, z) = (\tau_1 + \tau_2, x_2, 0, z_2)$. We consider the case that the point $(x_2, 0, z_2)$ exists in the hatched region illustrated in Fig. 9. In the figure, the boundary curves $R_1$ and $R_2$ are represented as following equations using parameters $u_1$ and $u_2$, respectively:

$$R_1: \begin{bmatrix} x \\ z \end{bmatrix} = F_2(-u_1) \times F_2^{-1}(0) \times \begin{bmatrix} 0 \\ Z_m \end{bmatrix}, \quad 0 \leq \omega_0 u_1 \leq \pi \quad (23)$$

$$R_2: \begin{bmatrix} x \\ z \end{bmatrix} = F_2(-u_2) \times F_2^{-1}(0) \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 0 \leq \omega_0 u_2 \leq \pi, -1 \leq z \leq Z_m.$$ Other boundaries are $L^+, L^-, B^-$, and $z$-axis ($z > Z_m$). If $(x_2, 0, z_2)$ exists in the above-mentioned region, the solution which returns back to $D^o$ starts to rotate divergently around $O$ constrained onto $D^o$ and hits $L^-$ at $(\tau, x, y, z) = (\tau_1 + \tau_2 + \tau_3, x_3, 0, Z_m)$. Therefore, a discrete one-dimensional map which transforms a point on $L^+$ into a point on $L^-$ can be defined:

$$f: L^+ \rightarrow L^-, x_0 \rightarrow x_3 = f(x_0) \quad (24)$$

where $x_0$ is the $x$-coordinate of the initial point on $L^+$ and $x_3 = f(x_0)$ is the $x$-coordinate of the point on $L^-$ which the solution leaving $L^+$ hits $L^-$. Since (10) is symmetric, we can define

$$\hat{f}: L^- \rightarrow L^+, x_0 \rightarrow \hat{f}(x_0) \quad (25)$$

where the map $\hat{f}$ transforms a point on $L^-$ into a point on $L^+$. The map $\hat{f}$ has the same form as $f$.

The concrete representations of $f$ are as follows.

1) $0 < x_0 \leq A$:

$$f(x_0) = x_3. \quad (26)$$

The value of $x_3$ is represented as follows:

$$x_3 = [1,0] \times F_2(\tau_0) \times F_2^{-1}(0) \times \begin{bmatrix} x_0 \\ -Z_m \end{bmatrix} \quad (27)$$
where $\tau_0$ is given by the following implicit equation:

$$Z_m = [0,1] \times F_2(\tau_0) \times F_2^{-1}(0) \times \begin{bmatrix} x_0 \\ -Z_m \end{bmatrix}, \quad 0 < \omega_0 \tau_0 < \pi.$$  \hspace{1cm} (28)

2) $A < x_0$: The solution leaving $L^+$ hits $B^-$ at $(\tau, x, y, z) = (\tau_1, x_1, 0, -1)$. The value of $x_1$ is represented as follows:

$$x_1 = [1,0] \times F_2(\tau_1) \times F_2^{-1}(0) \times \begin{bmatrix} x_0 \\ -Z_m \end{bmatrix} \hspace{1cm} (29)$$

where $\tau_1$ is given by the following implicit equation:

$$-1 = [0,1] \times F_2(\tau_1) \times F_2^{-1}(0) \times \begin{bmatrix} x_0 \\ -Z_m \end{bmatrix}, \quad 0 < \omega_0 \tau_1 < \pi.$$  \hspace{1cm} (30)

Next, the solution which enters $D^-$ hits $D^+$ at $(\tau, x, y, z) = (\tau_1 + \tau_2, x_2, 0, z_2)$. The values of $x_2$ and $z_2$ are represented as follows:

$$\begin{bmatrix} x_2 + X_p \\ z_2 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times F_1(\tau_2) \times F_1^{-1}(0) \times \begin{bmatrix} x_1 + X_p \\ Y_p \\ 0 \end{bmatrix} \hspace{1cm} (31)$$

where $\tau_2$ is given by the following implicit equation:

$$Y_p = [0,1] \times F_1(\tau_2) \times F_1^{-1}(0) \times \begin{bmatrix} x_1 + X_p \\ Y_p \\ 0 \end{bmatrix} \hspace{1cm} (32)$$

Next, the solution hits $L^-$ at $(\tau, x, y, z) = (\tau_1 + \tau_2 + \tau_3, x_3, 0, Z_m)$. The value of $x_3$ is represented as follows:

$$x_3 = [1,0] \times F_2(\tau_3) \times F_2^{-1}(0) \times \begin{bmatrix} x_2 \\ z_2 \end{bmatrix} \hspace{1cm} (33)$$

where $\tau_3$ is given by the following implicit equation:

$$Z_m = [0,1] \times F_2(\tau_3) \times F_2^{-1}(0) \times \begin{bmatrix} x_2 \\ z_2 \end{bmatrix}, \quad 0 < \omega_0 \tau_3 < \pi.$$  \hspace{1cm} (34)

At last, $f(x_0)$ is obtained as

$$f(x_0) = x_3.$$  \hspace{1cm} (35)

Hereafter, we fix $Z_m = 0.2$ and $\beta = 3$ and choose $\alpha$ as a bifurcation parameter. Let $\alpha_{\max}$ be the maximum value of $\alpha$ such that a one-dimensional map $f$ can be defined, that is, the point $(x, y, z) = (x_2, 0, z_2)$ exists in the hatched region in Fig. 9.

IV. GLOBAL BIFURCATIONS

Fig. 10 shows an example of $f$ obtained by calculating (26)–(35). The implicit equations are solved numerically. First the global bifurcations are discussed. We parameterize $f$ and $\hat{f}$ by the parameter $\alpha$ and write $f_{\alpha}$ and $\hat{f}_{\alpha}$ instead of $f$ and $\hat{f}$, respectively. $f_{\alpha}$ or $\hat{f}_{\alpha}$ has one extremum in $L^+$ or $L^-$. Let $Q$ be the point on $L^+$ at which $f_{\alpha}$ has an extremum and then $-Q$ is the point on $L^-$ at which $\hat{f}_{\alpha}$ has an extremum. Let $-x_{\max}$ be $f_{\alpha}(Q)$.

Hereafter we use a superscript "*" for an interval which is a part of $L^+$ and similarly we use "-" for an interval which is a part of $L^-$. Define:

$$\begin{align*}
J^+ &= \left[ f_{\alpha}(-x_{\max}), x_{\max} \right] \subset L^+, \\
J^- &= \left[ f_{\alpha}(x_{\max}), -x_{\max} \right] \subset L^-.
\end{align*}$$

(36)

In the following discussions, we consider the case $f_{\alpha}(J^+) \subset J^-$ is satisfied, then

$$\hat{f}_{\alpha}(J^-) \subset J^+.$$  \hspace{1cm} (37)

Define:

$$\hat{f}_{\alpha}(J^+) \subset J^+.$$  \hspace{1cm} (38)

From (37) and (38)

$$F_{\alpha} = f_{\alpha} \circ \hat{f}_{\alpha}: \quad L^+ \rightarrow L^+, \quad x_0 \rightarrow F_{\alpha}(x_0)$$

$$\hat{F}_{\alpha} = \hat{f}_{\alpha} \circ f_{\alpha}: \quad L^- \rightarrow L^-, \quad x_0 \rightarrow \hat{F}_{\alpha}(x_0).$$

(39)

From (37) and (38)

$$F_{\alpha}(J^+) \subset J^+,$$

$$\hat{F}_{\alpha}(J^-) \subset J^-.$$  \hspace{1cm} (40)

That is, $J^+$ (or $J^-$) is an invariant interval of $F_{\alpha}$ (or $\hat{F}_{\alpha}$).

We give the definition on a periodic point of $F_{\alpha}$.

Definition of Periodic Point

A point $k$ is said to be $m$-periodic point if $k = F_{\alpha}^m(k)$ and $k \neq F_{\alpha}^l(k)$ for $1 \leq l < m$. $k$ is said to be stable if $\left| DF_{\alpha}^m(k) \right| < 1$, where $DF_{\alpha}^m(k)$ denotes the differential coefficient of $F_{\alpha}^m$ at $k$.

Fig. 11 shows the one-parameter bifurcation diagrams on $L^+$. We take the initial condition $x_0 = Q$ in $J^+$ in Fig. 11(a) and $x_0 = -Q$ in $J^-$ in Fig. 11(b). We can find that there are differences between Figs. 11(a) and (b), which means that two distinct attractors coexist in this map. At these parameter values, two asymmetric attractors are located symmetrically with respect to the origin in the circuit.

We begin with the consideration of the phenomena in the range $\alpha \leq \alpha_1$ (see Fig. 11). Let $x_f$ be the point in $J^+$ such that $f_{\alpha}(x_f) = -x_f$, then $f_{\alpha}(-x_f) = x_f$, so that $F_{\alpha}(x_f) = x_f$. Therefore, $x_f$ is a fixed point of $F_{\alpha}$. For
Fig. 11. One-parameter bifurcation diagrams on $L^*$. (a) Initial condition is $x_0 = Q$. (b) Initial condition is $x_0 = -Q$.

Fig. 12. $f_a$ for $\alpha_{bl} < \alpha < \alpha_a$.

$\alpha < \alpha_{bl}, x_f$ is a stable fixed point, and the solution asymptotically approaches $x_f$. When $\alpha$ exceeds $\alpha_{bl}, x_f$ becomes unstable. At this parameter value, a symmetry-breaking transition occurs.

For $\alpha_{bl} < \alpha < \alpha_a$, $F_a^2(Q) < x_f$ is satisfied and then

\begin{align}
    f_a(J_a^+) &\subseteq J_a^- \\
    f_a(J_b^+) &\subseteq J_b^- \\
    \hat{f}_a(J_a^-) &\subseteq J_a^+ \\
    \hat{f}_a(J_b^-) &\subseteq J_b^+
\end{align}

(41)

where

\begin{align}
    J_a^+ &= [\hat{f}_a(-x_{\max}), x_f], J_b^+ = [x_f, x_{\max}] \\
    J_a^- &= [f_a(x_{\max}), -x_f], J_b^- = [-x_f, -x_{\max}]
\end{align}

(42)

(see Fig. 12). Therefore, for $\alpha_{bl} < \alpha < \alpha_a$:

\begin{align}
    F_a(J_a^+) &\subseteq J_a^+ \\
    F_a(J_b^+) &\subseteq J_b^+
\end{align}

(43)

and two distinct attractors exist in $J^*$: one passes through $J_a^+$ and $J_b^+$ alternately, and the other passes through $J_b^+$ and $J_a^+$ alternately. Since $F_a$ is like the logistic map [12] in the interval $J_a^*$ or $J_b^*$, $F_a$ is considered to exhibit period-doubling bifurcation route to chaos in each interval in this parameter range.

V. TWO TYPES OF WINDOWS

As shown in Section I, for $\alpha_{al} < \alpha < \alpha_{al}^*$ two interesting windows appear, which are the main objects of this paper. In this section we clarify the mechanisms and the structures of them.

First we give the accurate definition of a window.

Definition of Windows

It is said that a $m$-periodic window appears in some interval $[\alpha_a, \alpha_d]$, if there exist $\alpha_a, \alpha_w, \alpha_d$ such that:

1) At $\alpha = \alpha_a$, a tangent bifurcation occurs. $F_a$ has two $m$-periodic points near $Q$ for $\alpha_a < \alpha < \alpha_d$. At least for $\alpha_a < \alpha < \alpha_w$, one is a stable $m$-periodic point and the other is an unstable $m$-periodic point. Let the former be $x_+$ and also the latter be $x_-$.

2) At $\alpha = \alpha_w$, $x_+$ is equal to $Q$ (superstable).

3) At $\alpha = \alpha_d$, $F_a^2(Q)$ is equal to $x_+$. That is, the chaotic attractor in this window collides with the unstable $m$-periodic point $x_+$ and the chaotic regions widen to form a single band. This phenomena is called the interior crisis [21].

4) For $\alpha_a < \alpha < \alpha_d$, there exists the interval $J$ such that $F_a(J) \subseteq J$; $Q \notin J$ and $x_+ \notin J$, that is, an attractor of $F_a$ exists in this interval.

5) For $\alpha_a < \alpha < \alpha_d$, $F_a$ satisfies the Li–Yorke’s period 3 condition [22].

Proposition 1

$F_a$ satisfies the Li–Yorke’s period 3 condition for $\alpha_a < \alpha < \alpha_a$.

Proof: We define $X_{n+1}$ as follows.

\begin{align}
    X_{n+1} &= \begin{cases} f(X_n) \cdots X_n \in J^+ & n = 0, 1, 2, \ldots \\
    \hat{f}(X_n) \cdots X_n \in J^- & n = 0, 1, 2, \ldots
\end{cases}
\end{align}

(44)

For $\alpha_{al} < \alpha < \alpha_{al}^*$, $f_a$ or $\hat{f}_a$ is shown as Fig. 13 and there exists the point $X_0$ on $L^+$ such that the points $X_i$,
Consider the points $X_{-1}$, $X_{-2}$, $X_{-3}$, and $X_{-4}$. For $\alpha < a < a_{\max}$, these points can be chosen for which the following equations are satisfied:

\[ x_f < -x_{-1} < x_2 < -x_{-2} < x_{-3} < -x_1 < x_3 < x_{-4} < x_f < x_{-1}. \] (46)

where $x_f$ is a fixed point of $F_a$. Hence

\[ X_0 < -x_{-1} < -x_1 < -x_{-2} < x_f < x_{-3} < -x_1 < -x_{-3} < -x_2 < x_f. \] (47)

is satisfied. Since $X_{-2} = F_a(X_{-4})$, $X_0 = F_a^2(X_{-4})$, and $X_2 = F_a^3(X_{-4})$

\[ F_a^3(X_{-4}) > x_{-4} > F_a(X_{-4}) > F_a^2(X_{-4}) \] (48)

is satisfied.

Therefore, $F_a$ satisfies the Li–Yorke’s period 3 condition for $\alpha < a < a_{\max}$. Hence

There seems to exist infinitely many Type 1 and Type 2 windows for $\alpha < a < a_{\max}$. Examples of such windows are shown in Fig. 14. These figures are magnifications of the regions marked with $W_1$ and $W_2$ in Fig. 11 and associate with Fig. 5. At first, we explain the mechanism of the window of Type 1. Thereafter, we will explain the window of Type 2.

A. The Window of Type 1

We discuss the mechanism of the window of Type 1 as shown in Fig. 14(a), which is a magnification of the 3-periodic window in Fig. 3(g).

For the sake of the simplicity, we introduce some symbols. Let $Line-A^+$ and $Line-A^-$ be the lines which satisfy $f(x) = -x$ and $\tilde{f}(x) = -x$, respectively. Let $Line-B^+$ and $Line-B^-$ be the lines which satisfy $f_\alpha(x) = x$ and $\tilde{f}_\alpha(x) = x$, respectively.

Consider the composite maps $f_\alpha \circ \tilde{f}_\alpha \circ f_a$ and $f_\alpha \circ \tilde{f}_\alpha \circ f_\alpha$. The former transforms $J^+$ into $J^-$ and the latter transforms $J^-$ into $J^+$. $f_\alpha \circ \tilde{f}_\alpha \circ f_a$ in the neighborhood of $Q$ is shown in Fig. 15. The following situation is observed by computer calculation.

**Situation 1**

1) At $\alpha = \alpha_{a1}$, $f_\alpha \circ \tilde{f}_\alpha \circ f_a$ is tangent to the Line-$A^+$ in the neighborhood of $Q$. $F_a$ has two periodic points near $Q$ for $\alpha_{a1} < \alpha < a_{\max}$. At least for $\alpha_{a1} < \alpha < a_{a1}$ one is stable periodic point and the other is an unstable periodic point. Let the former be $x_{a1}$ and the latter be $x_{a1}$.

2) At $\alpha = a_{a1}$, $x_a = Q$. That is, $f_\alpha \circ \tilde{f}_\alpha \circ f_a(Q) = -Q$ is satisfied.

3) At $\alpha = a_{a1}$, $F_a^3(Q) = x_{a1}$. That is, the following conditions are satisfied.

4) Define the following intervals as shown in Fig. 15:

\[ J_a^+ = [x_{a1}, x_{a1}] \subset J^+ \quad J_a^- = [-x_{a1}, -x_{a1}] \subset J^- \]

For $\alpha_{a1} < \alpha < a_{a1}$

\[ f_\alpha \circ \tilde{f}_\alpha \circ f_a(J_a^+) \subset J_a^- \]

\[ \tilde{f}_\alpha \circ f_\alpha \circ \tilde{f}_\alpha(J_a^-) \subset J_a^+. \]

Then,

\[ F_a^3(J_a^+) \subset J_a^- \]

\[ \tilde{F}_a^3(J_a^-) \subset J_a^+. \]
Therefore, a 3-periodic window appears for $a_{a1} < a < a_{d2}$.

Note that the composite map $f_a \circ \hat{f}_a \circ f_a$ in $J_3^+$ has the similar form to the original map $f_a$ in $J^+$. Therefore, $f_a \circ \hat{f}_a \circ f_a$ in $J_3^+$ exhibits various phenomena observed in the original map $f_a$, such as, a symmetry-breaking transition and a symmetry-recovering crisis.

B. The Window of Type 2

Next, we discuss the mechanism of the window of Type 2 as shown in Fig. 14(b), which is a magnification of the 2-periodic window in Fig. 3(i).

Consider the composite maps $F_a^2 = f_a \circ \hat{f}_a \circ f_a$ and $\hat{F}_a^2 = f_a \circ \hat{f}_a \circ f_a \circ \hat{f}_a$. The former transforms $J^+$ into $J^+$ and the latter transforms $J^-$ into $J^-$. $F_a^2$ in the neighborhood of $Q$ is shown in Fig. 17. The following situation is observed by computer calculation.

Situation 2

1) At $a = a_{a2}$, $F_a^2$ is tangent to the Line-$B^+$ and a tangent bifurcation occurs. $F_a^2$ has two 2-periodic points near $Q$ for $a < a < a_{d2}$. At least for $a < a < a_{a1}$ one is a stable 2-periodic point and the other is an unstable 2-periodic point. Let the former be $x_{a2}$ and the latter be $x_{a2}$.  
2) At $a = a_{a2}$, $x_{a2} = Q$ (superstable).
3) At $a = a_{d2}$, $F_a^2(Q) = x_{a2}$, at which an interior crisis occurs.
4) For $a_{a2} < a < a_{d2}$, the following invariant intervals can be defined in $J^+$ as shown in Fig. 17: 

$$J_4^+ = [\hat{x}_{a2}, x_{a2}] \subset J^+, \quad J_4^- = [-\hat{x}_{a2}, -x_{a2}] \subset J^-$$

where 

$$F_a^2(\hat{x}_{a2}) = x_{a2}. \quad (52)$$

Therefore, a 2-periodic window appears for $a_{a2} < a < a_{d2}$.

For the purpose of simple explanation of the mechanism of this window, the maps $f_a$ and $\hat{f}_a$ at $a = a_{a2}$ are shown being overlapped in Fig. 18. We consider the orbit starting from $Q$ in $J^+$. The point $Q$ is transformed into a point in $J^-$ by $f_a$ (the points in $J^-$ are marked with $\bullet$ in Fig. 18). Thereafter, the point in $J^-$ is transformed into a point in $J^+$ by $\hat{f}_a$ (the points in $J^+$ are marked with $\star$ in Fig. 18). Fig. 18 clearly shows that the orbit starting from $Q$ makes the periodic attractor which returns to $Q$ through four periodic points in $J^-$ and $J^+$ alternately. Note that the positions of the periodic points in $J^+$ are different from those in $J^-$. Namely, in this case two distinct attractors coexist. Since $F_a^2: J_3^+ \to J_3^-$ is similar to the logistic map, it exhibits a period-doubling bifurcation route to chaos.

VI. SCALING MECHANISMS OF WINDOWS

In the previous section, it is shown that the similar structure to the global bifurcation diagram of $F_a$ is observed inside the window of Type 1. Actually, inside this window, the window of Type 1 and Type 2 appear infinitely many times. It means that in this circuit the number (one or two) and the nature (periodic or chaotic) are changed in an extremely complicated manner.

In this section in order to confirm this fact, we prove by using a simple scaling mechanism that there exists a sequence of the windows inside the window of Type 1 in which two types of windows appear alternately and infinitely many times.

As mentioned in the previous section, the composite map $f_a \circ \hat{f}_a \circ f_a$ makes the invariant interval $J_3^+$ near $Q$ for $a_{a1} < a < a_{d1}$. Define:

$$T_a = f_a \circ \hat{f}_a \circ f_a: J_3^+ \to J_3^-$$

Moreover, we rewrite $J_3^+ = [\hat{x}_{a1}, x_{a1}]$ as $I^+ = [l, m]$ and also $J_3^- = [-\hat{x}_{a1}, -x_{a1}]$ as $I^- = [-l, -m]$. In this section, we consider $T_a$ and $\hat{T}_a$ only for $a_{a1} < a < a_{d1}$.

Theorem 1
If $T_a$ satisfies the following (Condition), there exist the windows which appear in the following order:

Type 1:

Type 2:

$$n = 3, 4, 5, \ldots \quad (54)$$

where the numbers represent periods of the windows.
Condition

1) The Schwarzian derivative [23] of the map $T_\alpha$ is negative.
2) The map $T_\alpha$ has a unique extremum $Q$ in $I^+$.
3) As $\alpha$ increases continuously, the map $T_\alpha$ changes continuously from $T_{\alpha_1}$ to $T_{\alpha_2}$ as shown in Fig. 19 where $T_{\alpha_1}(x) = -x$ is satisfied for only $x = m$ and $T_{\alpha_2}(x) = -1$ is satisfied for only $x = Q$.

It is known that if the Condition 1 is satisfied, $T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots$ has at most one inflection point between its adjacent extrema. The above mentioned (Condition) is confirmed numerically.

Proof:

Fig. 20(a) and (b) show $T_{\alpha_1}: I^+ \rightarrow I^-$ and $T_{\alpha_1} \circ T_{\alpha_2}: I^+ \rightarrow I^+$ at $\alpha = \alpha_1$ (the symmetry recovering crisis). Define the interval $I^+_2$ as shown in Fig. 20(a):

$$I^+_2 = [l_2, m_2].$$

The following equations are satisfied.

$$|T_{\alpha_1}(T_{\alpha_2}(Q))|_{\alpha = \alpha_2} = m_2$$

(55)

$$|T_{\alpha_1} \circ T_{\alpha_2}(Q)|_{\alpha = \alpha_1} = m$$

(56)

$$DT_{\alpha_1}(T_{\alpha_2}(x)) =
\begin{cases}
< 0, & T_{\alpha_1}(T_{\alpha_2}(x)) < Q \\
> 0, & T_{\alpha_1}(T_{\alpha_2}(x)) > Q
\end{cases}$$

(57)

Let us consider the form of $T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_1}$ in $I^+_2$ for $\alpha_1 < \alpha < \alpha_2$ by the chain rule

$$D(T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_1}(x)) = DT_{\alpha_1}(T_{\alpha_2}(T_{\alpha_1}(x))) \cdot D(T_{\alpha_2}(T_{\alpha_1}(x))).$$

(60)

There exist the two points $e_1$ and $e_2$ ($l_2 < e_1 < Q < e_2 < m_2$) in $I^+_2$ such that $T_{\alpha_1}(T_{\alpha_2}(e_i)) = Q$ and

$$D(T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_1}(x)) =
\begin{cases}
< 0 \ldots l_2 < x < e_1 \quad \text{or} \quad Q < x < e_2 \\
> 0 \ldots e_1 < x < Q \quad \text{or} \quad e_2 < x < m_2
\end{cases}$$

(61)

The composite map $T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_1}$ has three extrema in $I^+_2$ as shown in Fig. 20(c).

From (55)–(57) the following equations are satisfied:

$$|T_{\alpha_1} \circ T_{\alpha_2}(Q)|_{\alpha = \alpha_2} = -l_2$$

(62)

$$|T_{\alpha_1} \circ T_{\alpha_2}(Q)|_{\alpha = \alpha_1} = -m$$

(63)

and for $\alpha_2 < \alpha < \alpha_1$:

$$T_{\alpha_1} \circ T_{\alpha_2}(e_1) = T_{\alpha_1} \circ T_{\alpha_2}(e_2) > T_{\alpha_1} \circ T_{\alpha_2}(Q).$$

(64)

Then, as $\alpha$ increases, there exists a process such that the neighborhood of $T_{\alpha_1} \circ T_{\alpha_2}(Q)$ ascends and intersects the Line $A^+$ at some parameter value. Moreover, if Condition 1 (negative Schwarzian derivative) is satisfied, $T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_1}$ touches the Line-$A^+$ in the neighborhood of $Q$ at a unique point (tangent bifurcation). Let the tangent bifurcation parameter value be $\alpha = \alpha_{a_3}$. Define the interval $I^+_3$ for $\alpha_3 < \alpha < \alpha_1$ as shown in Fig. 21(a):

$$I^+_3 = [l_3, m_3]$$

where

$$T_{\alpha_1} \circ T_{\alpha_1}(l_3) = T_{\alpha_1} \circ T_{\alpha_1}(m_3) = -l_3(l_3 < m_3).$$

(65)

Then, there exists a parameter value $\alpha_{a_3}(\alpha_{a_3} < \alpha_{a_3} < \alpha_1)$ such that

$$|T_{\alpha_1} \circ T_{\alpha_1}(Q)|_{\alpha = \alpha_{a_3}} = -m_3.$$ (66)

Hence, Situation 1 occurs in $T_{\alpha_1} \circ T_{\alpha_1} \circ T_{\alpha_1}$ in the neighbor-
hood of $Q$ for $\alpha_{d3} < \alpha < \alpha_{d4}$. Namely, the window of Type 1 appears for $\alpha_{d3} < \alpha < \alpha_{d4}$. Since $(T_{a} \circ T_{a})^{3}(l_{a}) = l_{a}$, the period of this window is $3 \times 3$ with respect to the original map $F_{a}$.

Next, we consider the form of the composite map $T_{a} \circ T_{a} \circ T_{a} \circ T_{a}$ in $I_{a}^{+}$ for $\alpha < \alpha_{d3}$. At $\alpha = \alpha_{d3}$, the interior crisis occurs and the above mentioned window disappears (see Fig. 21(a)). For $\alpha_{d3} < \alpha < \alpha_{d1}$, similarly, the following equation is satisfied:

$$D\{T_{a} \circ T_{a} \circ T_{a} \circ T_{a}(x)\}$$

$$= \begin{cases} 
0 & \text{for } l_{s} < x < e_{s} \\
Q & \text{for } Q < x < e_{d} \\
0 & \text{for } e_{d} < x < m_{3}
\end{cases} \quad (67)$$

where $e_{s}$ and $c_{s}(l_{s} < c_{s} < Q < e_{d} < m_{3})$ in $I_{a}^{+}$ satisfy $T_{a} \circ T_{a} \circ T_{a}(e_{s}) = Q(i = 3, 4)$. The composite map $T_{a} \circ T_{a} \circ T_{a} \circ T_{a}$ has three extrema in $I_{a}^{+}$ as shown in Fig. 21(b).

From (66), (63), and Fig. 20(a), the following equations are satisfied:

$$|T_{a} \circ T_{a} \circ T_{a}(Q)|_{\alpha = \alpha_{d3}} < l_{2} \quad (68)$$

$$|T_{a} \circ T_{a} \circ T_{a}(Q)|_{\alpha = \alpha_{d3}} = m \quad (69)$$

and for $\alpha_{d3} < \alpha < \alpha_{d1}$:

$$T_{a} \circ T_{a} \circ T_{a}(e_{s})$$

$$= T_{a} \circ T_{a} \circ T_{a}(e_{d}) < T_{a} \circ T_{a} \circ T_{a}(Q). \quad (70)$$

Then, as $\alpha$ increases, there exists a process such that the neighborhood of $T_{a} \circ T_{a} \circ T_{a}(Q)$ ascends and intersects the Line-$B^{+}$ at some parameter value. Let the tangent bifurcation parameter value be $\alpha = \alpha_{d4}$. Define the interval $I_{4}^{+}$ for $\alpha_{d4} < \alpha < \alpha_{d1}$ as shown in Fig. 22:

$$I_{4}^{+} = [l_{4}, m_{4}]$$

where

$$\hat{T}_{a} \circ T_{a} \circ T_{a}(l_{4}) = \hat{T}_{a} \circ T_{a}(m_{4}) = l_{4}(l_{4} < m_{4}). \quad (71)$$

Then, there exist a parameter value $\alpha_{d4}(\alpha_{d4} < \alpha < \alpha_{d4})$ such that

$$|\hat{T}_{a} \circ T_{a} \circ T_{a}(Q)|_{\alpha = \alpha_{d4}} = m_{4}. \quad (72)$$

Hence, Situation 2 occurs in $\hat{T}_{a} \circ T_{a} \circ T_{a}$ in the neighborhood of $Q$ for $\alpha_{d4} < \alpha < \alpha_{d4}$. Namely, the window of Type 2 appears for $\alpha_{d4} < \alpha < \alpha_{d4}$. Since $(T_{a} \circ T_{a})^{2}(l_{a}) = l_{a}$, the period of this window is $3 \times 2$ with respect to the original map $F_{a}$.

Note that the condition of $\hat{T}_{a} \circ T_{a} \circ T_{a}$ in $I_{4}^{+}$ (see Fig. 22) is the same as that of $T_{a} \circ T_{a}$ in $I_{4}^{+}$ (see Fig. 20(b)). Therefore, the above mentioned process can be applied to $\hat{T}_{a} \circ T_{a} \circ T_{a}$ in $I_{4}^{+}$ and we can repeat the similar discussion infinitely many times. As a result of this, we obtain Theorem 1. Q.E.D.

Fig. 23 shows some examples of the windows obtained by computer calculation, which appear in the order represented by (54). Initial condition is $x_{0} = Q$ in the above figures and it is $x_{0} = -Q$ in the below figures. (a) $3 \times 3$-periodic window of Type 1 (b) $3 \times 2$-periodic window of Type 2. (c) $3 \times 3$-periodic window of Type 1. (d) $3 \times 3$-periodic window of Type 2.
constant is obtained by calculating the following equation:

$$\delta(i) = \frac{\alpha_{w(i+1)} - \alpha_{w(i+2)}}{\alpha_{w(i+2)} - \alpha_{w(i+1)}}$$  (73)

where $\alpha_{w(i)}$ represents the parameter value at which $3 \times i$-periodic attractor for the window of Type 1 and $3 \times i/2$-periodic attractor for the window of Type 2 become superstable. The result obtained by calculating (73) with a computer is listed in Table I. It seems that $\delta(i)$ converges to 3.0654... when $i \to \infty$.

We could not get the accumulation point of the windows analytically. By numerical calculation $\alpha_{w(i)}$ seems to converge to $\alpha_{MI}$; the parameter value at which the interior crisis occurs.

**Remark**

It is known that the accumulation point of the sequence of the tangent bifurcations for the logistic map is the parameter value at which the interior crisis occurs [19], [24]. (For a two-dimensional map similar discussion has been done [25].)

Because our Poincaré map $f$ or $f^+$ is like the logistic map, a part of the theory on the successive tangent bifurcations for the logistic map may be used for our system. However, symmetry of the system is considered in our discussion and we show that the number of the attractors changes in a complicated manner. Moreover, we discuss the sequence of the tangent bifurcations based on the real physical system.

**VII. Conclusions**

In this paper, two types of the windows generated in a symmetric circuit have been investigated. By using a degeneration technique, we make clear the mechanisms of these two types of windows and prove rigorously that two types of the windows appear alternately and infinitely many times inside some windows under some reasonable assumptions by utilizing a certain scaling structure on the Poincaré map.

The one-dimensional Poincaré map derived from our circuit is like a combination of two logistic maps and it seems to be one of the simplest model of symmetric chaos-generating systems. Therefore, as the logistic map contributed to analysis of the Smale horseshoe [26], the results of this paper may contribute to the study of bifurcation phenomena in symmetric chaos-generating systems. To show the universality of our results may be our future problem.

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Toshimichi Saito, for a photograph and biography, please see page 409 of the March 1990 issue of this TRANSACTIONS.